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Compact, separable, linearly ordered spaces

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Abstract

A proof that a compact, separable, zero-dimensional, monotonically normal space is always a continuous image of a compact linearly ordered space is given. © 1998 Elsevier Science B.V.

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Monotonically normal spaces [2] are a natural class which includes all metric and linearly ordered spaces and have been studied extensively. See [1] for a recent survey.

There are several [11] outstanding unsolved questions about compact monotonically normal spaces, notably: *is every compact monotonically normal space*

(1) K_0 [3,4,10]?

(2) *acyclically monotonically normal* [4,10]?

(3) *the continuous image of a linearly ordered compact space* [5–7]?

A particular compact, monotonically normal space satisfying (3) has (2) and one satisfying (2) has (1). So we answer all three questions in the affirmative for the restricted class of zero-dimensional, separable spaces by proving:

Theorem. *If X is a zero-dimensional, separable, monotonically normal compactum, then X is a continuous image of a linearly ordered compactum.*

0. Outline of the proof

Since our proof is rather lengthy and full of notation and lemmas, we begin by outlining it:

We start by describing a "basic breakdown", taking full advantage of all four properties of X. By partitioning X into ever smaller pieces one arrives at what could be called

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the "atoms" of a particular breakdown. Lemma 2 gives the simple fact one gets out of such a breakdown, a fact used throughout the paper. Lemma 3 proves there is a basic breakdown whose atoms have no more than two points. (Separability is necessary for this lemma.)

We then fix a basic breakdown having atoms of cardinality at most two. Lemma 4 gives the basic property for $\mathcal{K}'$, the atoms of cardinality two at the nth level of our breakdown. We define what it means for a subset of $\mathcal{K}'$ to be "n-flat" and some of its properties are defined in Lemmas 5 and 6. Construction 8 is our basic construction, expanded to an $(\omega+1)$-sequence of such constructions in Section 7, with Lemmas 9 and 12 pointing out some properties of these constructions, while Construction 13 is an ω-sequence of constructions of the type of Construction 11 yielding a set Y and natural function f from Y onto X. In Section 9 a sequence of partial orders on Y is given whose limit, $\leqslant$, is proved in Section 10 to be a total order on Y. In Lemma 15 it is proved that Y, endowed with the $\leqslant$ order topology, is compact and various corollaries are stated. We begin Section 12 by proving several facts from which Lemma 17 showing that f is continuous can be proved.

This gives us the theorem.

Comment. Considerable work has been done on this subject, especially by J. Nikiel, but also by L.B. Treybig and S. Purisch, who have certainly proved special cases of parts of this proof in a different language and notation and they have interested others including me in this problem. It may be useful to the reader to look at the elementary example of S. Purisch in [7] to see a step one complication and its "cure": it shows a compact, zero-dimensional, separable, monotonically normal space which cannot be ordered but, by a simple operation, can be shown to be the continuous image of such a linearly ordered space.

1. Basic breakdowns

Our space X is monotonically normal and therefore [2] points are closed in X and X has an "MN operator" H. That is, for every closed A in X and open U with $A \subset U$, there is an open $H(A,U)$ with $A \subset H(A,U) \subset U$ such that:

(1) (normality) If $A \cap V = \emptyset$ and $B \cap U = \emptyset$, then $H(A,U) \cap H(B,V) = \emptyset$.

(2) (monotonicity) If $A \subset B$ and $U \subset V$, then $H(A,U) \subset H(B,V)$.

If $A = \{x\}$ for some $x \in X$, we simplify to just $H(x,U)$.

Definition 1. Let $\mathcal{W} = \{\text{nonempty } W \subset X \mid W \text{ is both open and closed}\}$. Let S be a countable dense subset of X. We call $\mathcal{F} = \bigcup\{\mathcal{F}_n \mid n \in \omega\}$ a "basic breakdown" of X provided:

(1) for each $n \in \omega$, $\mathcal{F}_n$ is a finite cover of X by disjoint members of $\mathcal{W}$, $\mathcal{F}_0 = \{X\}$, and, if $F \in \mathcal{F}_n$, $\mathcal{F}_{n+1}(F) = \{F' \in \mathcal{F}_{n+1} \mid F' \subset F\}$ is a cover of F;

(2) if $F \in \mathcal{F}_n$, then, for all $F' \in \mathcal{F}_{n+1}(F)$, there is $p \in F$ with $F' \subset H(p,F)$;

(3) if $p \neq q$ in S, there are disjoint F and F' in $\mathcal{F}$ with $p \in F$ and $q \in F'$. Also if $p \in S$ and $\{p\}$ is open, $\{p\} \in \mathcal{F}$.

Assuming $X \neq \emptyset$ there is a breakdown of X. Let

$$\mathcal{K} = \left\{\text{nonempty } K \subset X \mid \forall n \in \omega\ \exists F_n(K) \in \mathcal{F}_n \text{ such that } K = \bigcap_{n\in\omega} F_n(K)\right\}.$$

If $K \in \mathcal{K}$, let $c(K) = |K|$ if $|K| \leqslant 2$ and $c(K) = 3$ if $|K| \geqslant 3$. For each $i < c(K)$ choose W_i $(= W_i(K)) \in \mathcal{W}$ intersecting K such that $\{W_i \mid i < c(K)\}$ are disjoint and their union covers K. For each $i < c(K)$, let $K_i = W_i \cap K$. By compactness, there is some m $(= m(K)) \in \omega$ such that $F_m(K) \subset \bigcup\{H(K_i, W_i) \mid i < c(K)\}$. Define V_i $(= V_k(K)) = F_m(K) \cap H(K_i, W_i)$. Since the open $H(K_i, W_i)$, $i < c(K)$, are disjoint and $F_m(K) \in \mathcal{W}$, each $V_i \in \mathcal{W}$.

By Definition 1(3), if $K \in \mathcal{K}$ and $K \neq \{p\} \in \mathcal{W}$ for some $p \in S$, there is at most one point of S in K and every point of K is a limit point of S.

Lemma 2. *If $K \in \mathcal{K}$, $m(K) \leqslant m$, $L \in (\mathcal{K} - \{K\})$, and $F_m(L) = F_m(K)$, then $L \subset V_i(K)$ for some unique $i < c(K)$.*

Proof. Since $L \subset F_m(K)$, every point of L is in some $V_i(K)$. So assume $i < j < c(K)$ and both $V_i(K) \cap L \neq \emptyset$ and $V_j(K) \cap L \neq \emptyset$. Let n be minimal for $F_n(K) \neq F_n(L)$, $n > m$. By our definition of $\mathcal{F}_{n+1}$, for some $p \in F_n(L)$, $L \subset F_{n+1}(L) \subset H(p, F_n(L))$ (see Definition 1(2)). Since $n > m$, $p \in V_k(K)$ for some $k < c(K)$. But k is different from one of i and j, say $k \neq i$. Thus $p \notin W_i(K)$ and $V_i(K) \subset H(K_i, W_i(K))$. Since $K_i \subset K \subset F_n(K)$ and $F_n(K) \cap F_n(L) = \emptyset$, $K_i \cap F_n(L) = \emptyset$. Since H is an MN operator, $V_i(K) \cap H(p, F_n(L)) = \emptyset$ contrary to $V_i(K) \cap L \neq \emptyset$ and $L \subset F_{n+1}(L) \subset H(p, F_n(L))$. $\square$

2. Basic breakdowns with small atoms

Let $\mathcal{K}^* = \{K \in \mathcal{K} \mid c(K) = 3\}$. If $K \in \mathcal{K}^*$, since every point of K is a limit point of $S - K$, there is ℓ $(= \ell(K)) \in \omega$ such that $\ell > m(K)$ and, for all $i < 3$, $[H(K_i, V_i(K)) - F_\ell(K)] \neq \emptyset$.

Lemma 3. *Without loss of generality $\mathcal{K}^* = \emptyset$.*

Case 1. *Every basic breakdown yields a countable $\mathcal{K}^*$.*

Proof. For each countable ordinal α, by induction, we select a basic breakdown $\mathcal{F}(\alpha) = \bigcup\{\mathcal{F}_n(\alpha) \mid n \in \omega\}$ satisfying Definition 1. We then define $\mathcal{K}(\alpha)$ from this $\mathcal{F}(\alpha)$ exactly as we defined $\mathcal{K}$ from $\mathcal{F}$. We define $c(K)$ for $K \in \mathcal{K}(\alpha)$ by its cardinality exactly as before and let $\mathcal{K}^*(\alpha) = \{K \in \mathcal{K}(\alpha) \mid c(K) = 3\}$. By our construction which follows, $K \subset X$ can belong to $\mathcal{K}^*(\alpha)$ for at most one α, so for $K \in \mathcal{K}^*(\alpha)$ we choose $W_i(K)$ and $V_i(K)$ for $i < 3$ as well as $m(K)$ and $\ell(K)$ with reference to the $\mathcal{F}(\alpha)$ having K in $\mathcal{K}^*(\alpha)$.

Choose $\mathcal{F}(0)$ arbitarily (of course, satisfying Definition 1).

Suppose $\alpha = \beta + 1$ and all $\mathcal{F}_n(\beta)$ have been chosen. If $K \in \mathcal{K}^*(\beta)$, let

$$\mathcal{G}(K) = \{V_i(K) \mid i < 3\} \cup \Bigl\{X - \bigcup_{i<3} V_i(K)\Bigr\},$$

an open cover of X by four disjoint clopen sets. Since $\mathcal{K}^*(\beta)$ is countable by assumption we can index $\{\mathcal{G}(K) \mid K \in \mathcal{K}^*(\beta)\}$ as $\{\mathcal{G}_n(\beta) \mid n \in \omega\}$. Choose $\mathcal{F}_n(\alpha)$ so that it refines both $\mathcal{F}_n(\beta)$ and $\mathcal{G}_n(\beta)$ with $\{\mathcal{F}_n(\alpha) \mid n \in \omega\}$ again satisfying Definition 1.

Suppose α is a limit in ω_1 and that all $\mathcal{F}_n(\beta)$ have been chosen for all $\beta < \alpha$. Choose the $\mathcal{F}_n(\alpha)$ so that (in addition to conditions (1)–(3) in Definition 1):

(4) for $\beta < \alpha$ and $r \in \omega$, there is $n \in \omega$ such that $\mathcal{F}_n(\alpha)$ refines $\mathcal{F}_r(\beta)$, and

(5) for all $n \in \omega$ there are $\alpha_n < \alpha$ and $n_\alpha \in \omega$ such that $\mathcal{F}_n(\alpha) = \mathcal{F}_{n_\alpha}(\alpha_n)$.

To check that such a choice is possible for the limit α, suppose that such a choice has been possible for all limits $\gamma < \alpha$.

If $\alpha = \gamma + \omega$ for some limit γ for which conditions (1)–(5) are satisfied (or $\gamma = 0$), define $\alpha_n = \gamma + n$ for all n. Then choose $(n+1)_\alpha$ sufficiently large so that $\mathcal{F}_{(n+1)_\alpha}(\gamma + n + 1)$ refines both $\mathcal{F}_{(n_\alpha+1)}(\gamma + n)$ and $\mathcal{F}_n(\gamma + n + 1)$. Then $\mathcal{F}_n(\alpha) = \mathcal{F}_{n_\alpha}(\alpha_n)$ has all five properties. Clearly, (5) is automatic. Since $\mathcal{F}_{n_\alpha+1}(\gamma + n)$ refines $\mathcal{F}_{n_\alpha}(\gamma + n)$, we obtain Definition 1(1) and (2). Since for all $r \in \omega$, $\mathcal{F}_r(\gamma + n + 1)$ is a refinement of $\mathcal{F}_r(\gamma + n)$, having $\mathcal{F}_{(n+1)_\alpha}(\gamma + n + 1)$ refine $\mathcal{F}_n(\gamma + n + 1)$ ensures Definition 1(3) trivially and (4) because $\{\mathcal{F}_r(\gamma) \mid r \in \omega\}$ and $\{\mathcal{F}_r(\gamma + n + 1) \mid r \in \omega\}$ have these properties.

If α is a limit of an increasing sequence $\{\alpha_n \mid n \in \omega\}$ of limit ordinals with each $\{\mathcal{F}_r(\alpha_n) \mid r \in \omega\}$ having the five properties, we choose $(n+1)_\alpha > n_\alpha$ sufficiently large so that $\mathcal{F}_{(n+1)_\alpha}(\alpha_{n+1})$ refines $\mathcal{F}_{n_\alpha+1}(\alpha_r)$ for all $r \leqslant n$. Then $\{\mathcal{F}_n(\alpha) \mid n \in \omega\}$ has the desired properties if $\mathcal{F}_n(\alpha) = \mathcal{F}_{n_\alpha}(\alpha_n)$. □

Remark. The observations that one should make include:

(a) For $\alpha < \omega_1$, $\mathcal{K}(\alpha)$ is a partition of X into disjoint compact sets and, if $\beta < \alpha$, every member of $\mathcal{K}(\alpha)$ is a subset of some member of $\mathcal{K}(\beta)$.

(b) If $\beta < \alpha < \omega_1$, $L \in \mathcal{K}(\beta)$, $K \in \mathcal{K}^*(\alpha)$, and $K \subset L$, then $L \in \mathcal{K}^*(\beta)$ and, by our choice of the $\mathcal{F}_n(\beta + 1)$, any term of $\mathcal{K}(\beta + 1)$ intersecting L is contained in only one of $V_0(L)$, $V_1(L)$, or $V_2(L)$ and thus is a proper subset of L. By (a), K is a subset of some term of $\mathcal{K}(\beta + 1)$, so K is a proper subset of L.

(c) A trivial consequence of (a) and (b) is that any $K \in \bigcup_{\alpha<\omega_1} \mathcal{K}(\alpha)$ is in at most one $K^*(\alpha)$.

Assume the $\mathcal{F}_n(\alpha)$ as well as $\mathcal{K}(\alpha)$, $\mathcal{K}^*(\alpha)$, and $W_i(K)$, $V_i(K)$, $m(K)$, $\ell(K)$ for K in $\mathcal{K}^*(\alpha)$ have been selected.

Either, for some α, $\mathcal{K}^*(\alpha) = \emptyset$ and, taking $\mathcal{F} = \mathcal{F}(\alpha)$ we thus have Lemma 3, or, for all $\alpha < \omega_1$, $\mathcal{K}^*(\alpha) \neq \emptyset$. So assume $\mathcal{K}^*(\alpha) \neq \emptyset$ for any $\alpha < \omega_1$. We prove this is impossible and thereby prove Lemma 3.

If $\gamma < \omega_1$ and $F \in \mathcal{F}(\gamma)$, let

$$\mathcal{K}^*(F) = \bigl\{K \in \mathcal{K}^*(\alpha) \mid \alpha \text{ is a limit in } \omega_1 \text{ and } F = F_{\ell(K)}(K)\bigr\}.$$

Let

$$A_F = \{\text{limit } \alpha \text{ in } \omega_1 \mid \exists K \in (\mathcal{K}^*(F) \cap \mathcal{K}^*(\alpha))\}.$$

Since no $\mathcal{K}^*(\alpha) = \emptyset$, by (5), for each limit α there are $\gamma < \alpha$ and $F \in \mathcal{F}(\gamma)$ with $\alpha \in A_F$. By the pressing down lemma there is some $\gamma \in \omega_1$ and $F \in \mathcal{F}(\gamma)$ such that A_F is stationary in ω_1.

Claim. *There are infinitely many disjoint terms of $\mathcal{K}^*(F)$.*

Proof. Suppose otherwise. Since A_F is uncountable, for each $\tau < \omega_1$ there is a maximal nonempty family $\mathcal{M}_\tau$ of disjoint members of $\mathcal{K}^*(F)$ belonging to $\bigcup_{\alpha \geqslant \tau} \mathcal{K}^*(\alpha)$. By assumption $\mathcal{M}_\tau$ is finite and we choose each $\mathcal{M}_\tau$ having minimal cardinality among all such families for τ. Since the cardinality of the $\mathcal{M}_\tau$s can only increase, there is some τ_0 such that, for all $\tau > \tau_0$, $|\mathcal{M}_\tau| = |\mathcal{M}_{\tau_0}|$. Choose an uncountable $T \subset (\omega - \tau_0)$ such that $\sigma < \tau$ in T implies $\tau > \alpha$ for all $\alpha \in A_F$ with $\mathcal{M}_\sigma \cap K^*_\alpha \neq \emptyset$. Observe that, by the maximality of $\mathcal{M}_\sigma$, if $\sigma < \tau$ in T and $K \in \mathcal{M}_\tau$, there is $L \in \mathcal{M}_\sigma$ with $K \subset L$. For those n for which it is possible, choose $\sigma_n < \tau_n$ in T with $\tau_{n-1} < \sigma_n$, and $L_n \in \mathcal{M}_{\sigma_n}$ such that no term of $\mathcal{M}_{\tau_n}$ is contained in L_n. If $m < n$ and L_n is defined, then $L_n \subset K \in \mathcal{M}_{\tau_n}$ and $K \subset M \leqslant \mathcal{M}_{\sigma_m}$. Since $M \neq L_m$, by definition $L_m \cap L_n = \emptyset$. Since there do not exist infinitely many disjoint terms of $\mathcal{K}^*(F)$ by assumption, there is a maximal m for which L_m is defined; choose $\sigma > \tau_m$.

For all $\tau \in T$ with $\tau \geqslant \sigma$ choose $K_\tau \in \mathcal{M}_\tau$ with $K_\tau \subset K_\sigma$ for all $\tau > \sigma$. Again observe that if $\rho > \tau > \sigma$, $K_\rho \subset L \in \mathcal{M}_\tau$ and $L \subset M \in \mathcal{M}_\sigma$; but $K_\rho \subset K_\sigma$ so $M = K_\sigma$. But $L = K_\tau$ since for each term of $K \in \mathcal{M}_\sigma$ there can only be one term $\mathcal{M}_\tau$ contained in K since $|\mathcal{M}_\sigma| = |\mathcal{M}_\tau|$. Thus $\{K_\tau \mid \tau \in T,\ \tau \geqslant \sigma\}$ is an uncountable, strictly decreasing by inclusion sequence of compact nonempty subsets of the compact separable monotonically normal space X. In [8] Ostaszewski proves that closed sets are G_δ sets in such a space so the existence of the K_τs is impossible. □

For all $t \in \omega$ choose $K_t \in \mathcal{K}^*(F)$ such that $\{K_t \mid t \in \omega\}$ are disjoint. Say $K_t \in \mathcal{K}(\alpha(t))$. We assume $s < t$ implies $\alpha(s) < \alpha(t)$ and all $\alpha(t)$ are in A_F.

Suppose $s < t$. There is $K \in \mathcal{K}^*(\alpha(s))$ with $K_t \subset K$. Since $K \neq K_s$, $K \cap F \neq \emptyset$, $K \subset F = F_\ell(K_s)$ and $m(K_s) < \ell(K_s)$, by Lemma 2, there is $i < 3$ with $K \subset V_i(K_s)$; thus $K_t \subset V_i(K_s)$.

There is $n \in \omega$ such that $F_n(K_s) \neq F_n(K)$. Since $\alpha(t)$ is a limit, by (4), there is $r \in \omega$ such that $\mathcal{F}_r(\alpha(t))$ refines $\mathcal{F}_n(\alpha(s))$, so $K_t \subset F_r(K_t) \subset F_n(K)$. There is $x \in F_n(K_s)$ such that $K_s \subset F_{n+1}(K_s) \subset H(x, F_n(K_s))$. Since $m(K_t) < \ell(K_t)$, $x \in V_j(K_t)$ for some unique $j < 3$. If $k < 3$ and $k \neq j$, then $V_k(K_t) \cap K_s = \emptyset$ because H is an MN operator and $V_k(K_t) = H((K_t)_k, W_k(K_t))$, $x \notin W_k(K_t)$, $(K_t)_k \subset K_t \subset F_n(K)$ which does not intersect $F_n(K_s)$ and $K_s \subset H(x, F_n((K_s))$. Since every point of K_s is in $V_h(K_t)$ for some $h > 3$, that h is j. Thus $K_s \subset V_j(K_t)$.

Ramsey's [9] theorem $\omega \to \omega^2_r$ states that if $r \in \omega$ and the pairs in ω are partitioned into r sets, there is an infinite $T \subset \omega$ all of whose pairs are in one set.

Partition the pairs $s \neq t$ from ω into sets $\{I_{ij} \mid i < 3,\ j < 3\}$ where $\{s,t\} \in I_{ij}$ if $s < t$, $K_t \subset V_i(K_s)$, and $K_s \subset V_j(K_t)$. By Ramsey's theorem there are $i < 3$, $j < 3$, and an infinite $T \subset \omega$ such that, for all $s < t$ in T, $\{s,t\} \in I_{ij}$. Without loss of generality we can assume $i = 1$ and $j = 2$. Again using the fact that H is an MN operator for X, for all $s < t$ in T, $V_0(K_t) \cap V_0(K_s) = \emptyset$. For $V_0(K_t) = H((K_t)_0, W_0(K_t))$, $V_0(K_s) = H((K_s)_0, W_0(K_s))$, $(K_t)_0 \subset K_t \subset V_1(K_s)$ which misses $W_0(K_s)$, and $(K_s)_0 \subset K_s \subset V_2(K_t)$ which misses $W_0(K_t)$. Thus $\{V_0(K_t) \mid t \in T\}$ are disjoint.

For each $t \in T$ define $U_t = H((K_t), V_0(K_t))$; $\overline{U}_t \subset V_0(K_t)$. Since $F = F_{\ell(K_t)}(K_t)$, by the definition of ℓ, $(U_t - F) \neq \emptyset$. Since X is compact and F is open, there is

$$p \in \left(\overline{\bigcup\{U_t \mid t \in T\}} - \left(F \cup \bigcup_{t \in T} \overline{U}_t\right)\right).$$

Since p can belong to $V_0(K_t)$ for at most one t, we can choose t so that $p \notin V_0(K_t)$ while $U_t \cap H(p, X - F) \neq \emptyset$. Since $p \notin V_0(K_t)$ and $U_t = H((K_t)_0, V_0(K_t))$ while $K_t \subset F$, this contradicts H being an MN operator.

Case 2. *There is a basic breakdown $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ such that $\mathcal{K}^*$ is uncountable.*

Proof. This is proved impossible in [6] but we also give a proof for completeness.

Since $\mathcal{F}$ is countable, there are $F \in \mathcal{F}$ and $\{K_t \mid t \in \omega\} \subset \mathcal{K}^*$ such that $F = F_\ell(K_t)$ for all $t \in \omega$. The members of $\mathcal{K}^*$ are disjoint. Since $m < \ell$, by Lemma 2, for all $s \neq t$ in ω, there is $i < 3$ with $K_t \subset V_i(K_s)$ (and $j < 3$ with $K_s \subset V_j(K_t)$). The proof given in the preceding two paragraphs thus leads to a contradiction here also. Which completes our proof of Lemma 3. $\square$

3. The basic property of atoms

We now return to the situation described before Lemma 2: we have a simple fixed basic breakdown $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, satisfying Definition 1, a resulting $\mathcal{K}$ and, by Lemma 3, we assume that each $K \in \mathcal{K}$ has at most two members. We choose $\mathcal{F}_0 = \{X\}$. For the rest of the paper we shall primarily be concerned with $\mathcal{K}' = \{K \in \mathcal{K} \mid |K| = 2\}$. To simplify notation for $K = \{x, y\} \in \mathcal{K}'$ where $K_0 = \{x\}$ and $K_1 = \{y\}$ we just write $K_0 = x$ and $K_1 = y$. We use $i \neq i'$, $j \neq j'$, and $k \neq k'$ for the integers 0 and 1.

For $n \in \omega$ let $\mathcal{K}'_n = \{K \in \mathcal{K}' \mid m(K) \leqslant n\}$ and for $C \subset X$ we let $\mathcal{K}_n(C) = \{K \in \mathcal{K}'_n \mid K \subset C\}$. Fix some $n \in \omega$ and $F \in \mathcal{F}_n$ until further notice.

If $K \neq K'$ in $\mathcal{K}_n(F)$ we say that K_i and K'_j *link* if $K \subset V_j(K')$ and $K' \subset V_i(K)$. If K and K' are in $\mathcal{K}_n(F)$, then by Lemma 2 there are i and j for which K_i and K'_j link and in this case $W_{i'}(K) \cap W_{j'}(K') = \emptyset$.

Lemma 4. *Suppose K, K' and K'' are in $\mathcal{K}_n(F)$. Then K_i and K'_j link and $K'_{j'}$ and K''_k link imply K_i and K''_k link.*

Proof. Without loss of generality we assume $i = j = 0$ and $j' = k = 1$. Since K_0 and K'_0 link, $K' \subset V_0(K) \subset H(K_0, W_0(K))$. Since K'_1 and K''_1 link, $K' \subset V_1(K'') \subset$

$H(K_1'', W_1(K''))$. Thus since H is an MN operator either $(K_0 \cap W_1(K'')) \neq \emptyset$ or $(K_1'' \cap W_0(K)) \neq \emptyset$.

Suppose $(K_1'' \cap W_0(K)) \neq \emptyset$. Then by Lemma 2 $K_1'' \subset K'' \subset V_0(K)$. To prove that K_0 links K_1'' it remains to show that $K \subset V_1(K'')$. Otherwise $K \subset V_0(K'') \subset H(K_0'', W_0(K''))$. Since K_0 and K_0' link, $K \subset V(K_0') \subset H(K_0', W_0(K'))$. So either $(K_0' \cap W_0(K'')) \neq \emptyset$ or $(K_0'' \cap W_0(K')) \neq \emptyset$. Since K_1' and K_1'' link, $K' \subset V_1(K'')$ so we must have $(K_0'' \cap W_0(K')) \neq \emptyset$. But this contradicts $K_0'' \subset K'' \subset V_1(K')$ which follows from K_1' linking K_1''.

A symmetric argument shows that $K_0 \cap W_1(K'') \neq \emptyset$ implies that K_0 and K_1'' link. $\square$

4. n-flat sets

We say $\mathcal{L}$ is n-flat if $\mathcal{L} \subset \mathcal{K}_n(F)$ for some $F \in \mathcal{F}_n$ and there is a total order $\leqslant$ on $\bigcup \mathcal{L}$ such that $K \neq K'$ in $\mathcal{L}$, K_i links K_j', and $K_{i'} < K_i$ imply $K_{i'} < K_i < K_j' < K_{j'}'$ while $K_{i'}$ links $K_{j'}'$ implies $K_j' < K_{j'}' < K_{i'} < K_i$. We might say that $K_{i'} < K_i$ induces these orders, the same one induced by $K_j' < K_{j'}'$. Thus $K_{i'} < K_i$ induces *all* of $\leqslant$ and there are only two n-flat orders on $\bigcup \mathcal{L}$, the one induced by $K_i < K_{i'}$ and the one induced by $K_{i'} < K_i$.

Observe that $\emptyset$ is n-flat for all $n \in \omega$.

Fix some nonempty n-flat set $\mathcal{L} \subset \mathcal{K}_n(F)$ and n-flat order $\langle \bigcup \mathcal{L}, \leqslant \rangle$. If $K \in (\mathcal{K}_n(F) - \mathcal{L})$, let $\mathcal{L}_i(K) = \{L \in \mathcal{L} \mid L \subset V_i(K)\}$. If $\bigcup \mathcal{L}_0(K)$ and $\bigcup \mathcal{L}_1(K)$ partition $\bigcup \mathcal{L}$ into disjoint $\leqslant$ intervals, $\langle \bigcup \mathcal{L}, \leqslant \rangle$ can be extended to $\langle (\bigcup \mathcal{L} \cup K), \leqslant \rangle$ and $\mathcal{L} \cup \{K\}$ is n-flat. Otherwise there is no n-flat order on $(\bigcup \mathcal{L}) \cup K$. Thus:

Lemma 5. *If $(\mathcal{L} \cup \{K\})$ is not n-flat, then $(\bigcup \mathcal{L}) \subset V_i(K)$ for some i.*

Proof. Without loss of generality $L_0 < L_1$ for all $L \in \mathcal{L}$. Suppose K_i links L_1 for some $L \in \mathcal{L}$. Then, by Lemma 4, K_i links L_1' for all $L_1' \leqslant L_1$ in $\langle \bigcup \mathcal{L}, \leqslant \rangle$; so $K_{i'}$ does *not* link M_1 for any $M \in \mathcal{L}$. Similarly, if K_i links L_0 for some $L \in \mathcal{L}$, then K_i links L_0' for all $L_0 \leqslant L_0'$ in $\langle \bigcup \mathcal{L}, \leqslant \rangle$ and $K_{i'}$ does not link M_0 for any $M \in \mathcal{L}$. Thus either $\mathcal{L} = \mathcal{L}_i(K)$ for some $i < 2$ (so $\bigcup \mathcal{L} \subset V_i(K)$), or $\bigcup \mathcal{L}_0(K)$ and $\bigcup \mathcal{L}_1(K)$ partition $\bigcup \mathcal{L}$ into disjoint $\leqslant$ intervals (and thus $\mathcal{L} \cup \{K\}$ is n-flat). $\square$

5. More properties of n-flat sets and orders

Lemma 6. *$\emptyset \neq \mathcal{L}' \subset \mathcal{L}$ and $A = \bigcup \{V_{i'}(K) \mid K \in \mathcal{L}'$ and $K_{i'} < K_i\}$ then either $\langle \bigcup \mathcal{L}', \leqslant \rangle$ has a maximal element K_i and $A = V_{i'}(K)$ or there is a unique $a \in (\overline{A} - A)$ with $a \in \overline{\bigcup \mathcal{L}'}$.*

Proof. Assume, without loss of generality, that $K_0 < K_1$ for all $K \in \mathcal{L}$. If $K_1' < K_0$ in $\langle \bigcup \mathcal{L}', \leqslant \rangle$ then $K_0' < K_1' < K_0 < K_1$. If $\langle \bigcup \mathcal{L}', \leqslant \rangle$ has a maximal element K_i, then $i = 1$, $A = V_0(K)$ which is clopen, and $(\bigcup \mathcal{L}' - \{K_1\}) \subset A$. So assume $\langle \bigcup \mathcal{L}', \leqslant \rangle$ has no maximal element. Since $K_1' < K_0$ implies $(K' \cup V_0(K')) \subset V_0(K)$, but $K_1 \notin V_0(K)$ and $(\bigcup \mathcal{L}') \subset A$, by the compactness of F, there must be some

$$x \in \bigcap \left\{ \overline{J} \mid J \text{ is a terminal interval of } \left\langle \bigcup \mathcal{L}', \leqslant \right\rangle \right\}$$

and $x \notin A$. Suppose there were $x \neq x'$ in $\overline{A} - A$.

Let U and U' be disjoint clopen sets with $x \in U$ and $x' \in U'$. Without loss of generality $K_1' < K_0$ for some K and K' in $\mathcal{L}'$ such that $H(x', U') \cap V_0(K') \neq \emptyset$ and $H(x, U) \cap V_0(K) \neq \emptyset$. But $V_0(K') \subset V_0(K)$ which contradicts H being an MN operator and $U' \cap U = \emptyset$. $\square$

Analogously:

Lemma 7. *If $Z = \bigcup \{V_j(K) \mid K \in \mathcal{L}'$ and $K_{j'} < K_j\}$, then either $\langle \bigcup \mathcal{L}', \leqslant \rangle$ has a minimal element $K_{j'}$ and $Z = V_j(K)$ or there is a unique $z \in (\overline{Z} - Z)$.*

6. The basic construction

Construction 8. Suppose $n \in \omega$, C is a separable, compact subset of $F \in \mathcal{F}_n$, $F^* \in \mathcal{F}$, and $\langle \bigcup \mathcal{L}, \leqslant \rangle$ is a maximal n-flat ordered set from $\mathcal{K}_n(C)$. We construct $\mathcal{M}$, $\mathcal{M}^*$, $\mathcal{B}$ and $\mathcal{D}$, as well as a_M, z_M, $\mathcal{B}_M$ and Q_M for $M \in \mathcal{M}$, a total order $\leqslant$ on $\mathcal{M}$, and $\mathcal{L}_B$, $\leqslant_B$, q_B, and B^* for $B \in \mathcal{B}$.

If $\mathcal{K}_n(C) = \emptyset$, $\mathcal{L} = \emptyset$. Take $\mathcal{B} = \emptyset$ and $\mathcal{M} = \mathcal{M}^* = \mathcal{D} = \{C\}$ while other things remain undefined. Now assume $\mathcal{K}_n(C) \neq \emptyset$.

For all $x \in C$, let

$$A_x = \bigcup \left\{ V_{i'}(K) \mid K \in \mathcal{L},\ K_{i'} < K_i \text{ and } x \in V_i(K) \right\},$$

$$Z_x = \bigcup \left\{ V_i(K) \mid K \in \mathcal{L},\ K_{i'} < K_i \text{ and } x \in V_{i'}(K) \right\},$$

and $M_x = C - (A_x \cup Z_x)$. Since $A_x \cap (\bigcup \mathcal{L})$ and $Z_x \cap (\bigcup \mathcal{L})$ are disjoint initial and terminal intervals of $\langle \bigcup \mathcal{L}, \leqslant \rangle$, if $M_x \neq M_y$, $M_x \cap M_y = \emptyset$. Since A_x and Z_x are open, $\mathcal{M} = \{M_x \mid x \in C\}$ is a family of disjoint compact sets. Define $M_x \leqslant M_y$ in $\mathcal{M}$ if $(A_x \cap (\bigcup \mathcal{L})) \subset (A_y \cap (\bigcup \mathcal{L}))$.

Suppose $M = M_x \in \mathcal{M}$. If $A_x \neq \emptyset$, by Lemma 6, either $A_x \cap (\bigcup \mathcal{L})$ has a maximal element $K_{i'}$, in which case $A_x = V_{i'}(K)$ and $K_i \in M$, or there is a unique $a_M \in (\overline{A}_x - A_x)$ with $a_M \in M$. Similarly, by Lemma 7, either $Z_x \cap (\bigcup \mathcal{L})$ has a minimal element K_i, in which case $Z_x = V_i(K)$ and $K_{i'} \in M$, or there is a unique $z_M \in (\overline{Z}_x - Z_x)$ with $z_M \in M$. Let

$$\mathcal{M}^* = \left\{ M \in \mathcal{M} \mid M \not\subset \{a_M, z_M\} \text{ and } |M| > 1 \right\}.$$

Since each $(M - \{a_M, z_M\})$ is open in the separable C, $\mathcal{M}^*$ is countable and each $M \in \mathcal{M}^*$ is separable.

By Lemma 5 and the maximality of $\langle \mathcal{L}, \leqslant \rangle$, if $K \in (\mathcal{K}_n(C) - \mathcal{L})$ there is $i < 2$ with $(\bigcup \mathcal{L}) \subset V_i(K)$. At least for the remainder of this construction, reindex $K_n(C)$ so that $K \in (\bigcup \mathcal{L})$ implies $K_0 < K_1$ and, if $K \in (\mathcal{K}_n(C) - \mathcal{L})$, $(\bigcup \mathcal{L}) \subset V_0(K)$.

Suppose $M \in \mathcal{M}^*$. For all $\alpha < 2^{|X|}$ for which it is possible, choose an open V_α by induction as follows. If possible, choose a nonempty maximal from $\mathcal{K}_n(M) - (\bigcup_{\beta<\alpha} V_\beta)$ with $K_0 <_\alpha K_1$ for all $K \in \mathcal{L}_\alpha$, n-flat ordered set $\langle \mathcal{L}_\alpha, \leqslant_\alpha \rangle$. If there is a term of $\mathcal{K}_n(M) - (\bigcup_{\beta<\alpha} V_\beta)$ contained in F^*, make sure $\mathcal{L}_\alpha$ has such a term. Define $V_\alpha = \bigcup\{V_1(K) \mid K \in \mathcal{L}_\alpha\}$ and $B_\alpha = C \cap V_\alpha$. If $\kappa = \{\alpha \mid V_\alpha$ is defined$\}$ let $\mathcal{B}_M = \{B_\alpha \mid \alpha < \kappa\}$ and, if $B = B_\alpha$, let $\mathcal{L}_B = \mathcal{L}_\alpha$ and $\leqslant_B = \leqslant_\alpha$.

Suppose $\beta < \alpha < \kappa$ and $L \in \mathcal{L}_\alpha$. Since $L \not\subset B_\beta$, for every $K \in \mathcal{L}_\beta$ either L_0 or L_1 links K_0. By the maximality of $\mathcal{L}_\beta$, L_1 does not link K_0 for all $K \in \mathcal{L}_\beta$ so L_0 links K_0 for some $K \in \mathcal{L}_\beta$. By Lemma 4 then, L_0 links K_0' for all $K_0 \leqslant_\beta K_0'$ in $\mathcal{L}_\beta$. If L_1 links K_0'' for some $K'' \in \mathcal{L}_\beta$, then L_1 links K_0' for all $K_0'' \leqslant_\beta K_0'$. But $K_0'' \leqslant_\beta K_0$ or $K_0 \leqslant_\beta K_0''$ so this is impossible. Thus $\beta < \alpha < \kappa$ implies K_0 links L_0 for *all* $K \in \mathcal{L}_\beta$ and $L \in \mathcal{L}_\alpha$ and $B_\alpha \cap B_\beta = \emptyset$. Since $\mathcal{B}_M$ is a family of disjoint sets which are open in the separable C, $\mathcal{B}_M$ is countable and its members are separable.

For each $B \in \mathcal{B}_M$, by Lemma 6, either there is a minimal $K_0 \in \mathcal{L}_B$, in which case $K_0 = q_B \in (C - B)$ and $B = V_1(K) \cap C$, or there is a unique $q_B \in (\overline{B} - B)$. Define $Q_M = \{q_B \mid B \in \mathcal{B}_M\}$. Observe that $Q_M \subset M$ and $(\bigcup \mathcal{B}_M) \subset M$. If $x = q_B = K_0 \in (C - B)$, $K \subset M$ and $(\bigcup \mathcal{L}) \subset V_0(K)$; so $V_1(K) \cap (A_x \cup Z_x) = \emptyset$, $(\{q_B\} \cup B) \subset M$, and B is clopen in C. If $\{q_B\} = (\overline{B} - B)$, $q_B \in \overline{\bigcup \mathcal{L}_B}$. Since $(\bigcup \mathcal{L}_B) \subset M$ which is compact, $q_B \in M$ and, if $L \in \mathcal{L}_B$ and $x = L_0 \in M$, $V_1(L) \cap (A_x \cup Z_x) = \emptyset$. So again $(\{q_B\} \cup B) \subset M$ and $\overline{B} = B \cup \{q_B\}$.

Let $\mathcal{B} = \bigcup\{\mathcal{B}_M \mid M \in \mathcal{M}^*\}$; $\mathcal{B}$ is a countable family of disjoint separable, open in C, sets. Let $B^* = (B \cup \{q_B\})$ for $B \in \mathcal{B}$; B^* is compact.

For $M \in \mathcal{M}^*$, define $D_M = M - (\bigcup \mathcal{B}_M)$. One consequence of Lemma 9 below is that $D_M - \overline{Q}_M$ is open in M and thus, since Q_M is countable and M is separable and compact, D_M is separable and compact. Hence $\mathcal{D} = \{D_M \mid M \in \mathcal{M}^*\}$ is a countable set of disjoint compact separable subsets of C.

We call the $\mathcal{M}$, $\mathcal{M}^*$, $\mathcal{B}$ and $\mathcal{D}$ we have just constructed, a "type 8" construction for $\langle F^*, n, C, \langle \bigcup \mathcal{L}, \leqslant \rangle \rangle$. We could say $\mathcal{M} = \mathcal{M}(\langle F^*, n, C, \langle \bigcup \mathcal{L}, \leqslant \rangle \rangle)$, $\mathcal{M}^* = \cdots$, but the notation is defeating.

Lemma 9. *If $\mathcal{B}' \subset \mathcal{B}$, then $x \in (\overline{\bigcup \mathcal{B}'} - \bigcup \mathcal{B})$ only if $x \in \overline{\{q_B \mid B \in \mathcal{B}'\}}$.*

Proof. Suppose otherwise. Since $C = (\bigcup \mathcal{M})$, $x \in M_x \in \mathcal{M}$. Let

$$\mathcal{A} = \{B \in \mathcal{B}' \mid B \in \mathcal{B}_M \text{ for some } M \subset A_x\}.$$

If $x \in \overline{\bigcup \mathcal{A}}$, $x = a_{M_x}$ and $\mathcal{M}' = \{M \in \mathcal{M}^* \mid \mathcal{B}_M \cap \mathcal{A} \neq \emptyset\}$ is cofinal in $\{M'' \in \mathcal{M} \mid M'' < M_x \text{ in } \langle \mathcal{M}, \leqslant \rangle\}$. For each $M \in \mathcal{M}'$ choose $B_M \in (\mathcal{B}_M \cap \mathcal{A})$. Then

$$x \in \overline{\{q_{B_M} \mid M \in \mathcal{M}'\}}.$$

Similarly if

$$\mathcal{Z} = \{B \in \mathcal{B}' \mid B \in \mathcal{B}_M \text{ for some } M \in \mathcal{M}^*,\ M \subset Z_x\}$$

and $x \in \overline{\bigcup \mathcal{Z}}$, then $x \in \overline{\{q_B \mid B \in \mathcal{Z}\}}$. So without loss of generality, $\mathcal{B}' \subset \mathcal{B}_{M_x}$.

Suppose U is a clopen neighborhood of x missing $\{q_B \mid B \in \mathcal{B}'\}$. Since $x = L_i \in L \in \mathcal{K}$ (of course L can be $\{x\}$), we can assume $U = (F_m(L) \cap V_i(L))$ for some $m > m(L)$ and $m > n$.

If $B \in \mathcal{B}'$, $q_B = K_0$, and $B = V_1(K)$, then $B \cap H(x, U) = \emptyset$. For $x \notin B$ implies $x \notin V_1(K)$ and $F \subset V_0(K) \cup V_1(K)$ implies $x \in V_0(K)$. So $x \notin W_1(K)$. Since H is an MN operator and $V_1(K) = H(K_1, W_1(K))$, if $B \cap H(x,U) \neq \emptyset$, $K_1 \in U$. By Lemma 2 and our choice of U, since $K_1 \in U$, $K \subset U$ and $q_B = K_0 \in U$ contrary to assumption. Thus we can assume that $\mathcal{B}' \subset \{B \in \mathcal{B}_{M_x} \mid q_B \in \overline{B}\}$.

There is $B \in \mathcal{B}'$ and $L \in \mathcal{L}_B$ such that $V_1(L) \cap H(x, U) \neq \emptyset$. Since $q_B \in (X - U)$ which is clopen there is $K \in \mathcal{L}_B$ such that $K <_B L$ in $\langle \mathcal{L}_B, \leqslant_B \rangle$ with $K \subset (X - U)$. Since $x \notin V_1(K) \subset B$, $x \in V_0(K)$ and $x \notin W_1(K)$. Since $V_1(L) \subset V_1(K)$ we have a contradiction to H being an MN operator since $H(K_1, W_1(K)) \cap H(x, U) \neq \emptyset$ but $K_1 \notin U$ and $x \notin W_1(K)$. $\square$

Corollary 10. *If $\mathcal{B}' \subset \mathcal{B}$ and $\{q_B \mid B \in \mathcal{B}'\} \subset U$ which is clopen in X, there are at most finitely many $B \in \mathcal{B}'$ such that $B \not\subset U$.*

7. The sequence of constructions

Construction 11. Suppose $n \in \omega$, C is a separable compact subset of $F \in \mathcal{F}_n$ and $\langle \bigcup \mathcal{L}, \leqslant \rangle$ is a maximal n-flat ordered set from $\mathcal{K}_n(C)$. For all $t \in \omega$ we construct $\mathcal{M}_t(C)$, $\mathcal{M}_t^*(C)$, $\mathcal{B}_t(C)$, $\mathcal{B}_t^*(C)$, and $\mathcal{D}_t(C)$. (These depend on n and $\langle \bigcup \mathcal{L}, \leqslant \rangle$ as well as on C.) We also define $\mathcal{D}_\omega(C)$ and q_D for each $D \in \mathcal{D}_\omega(C)$.

Let $\{F_t \mid t \in \omega\}$ be any indexing of $\mathcal{F}$. Define $\mathcal{B}_0 = \{C\}$ and let $\mathcal{M}(C)$, $\mathcal{M}^*(C)$, $\mathcal{B}(C)$ and $\mathcal{D}(C)$ be a "type 8" construction for $\langle F_0, n, C, \langle \bigcup \mathcal{L}, \leqslant \rangle \rangle$. Then define $\mathcal{M}_0(C) = \mathcal{M}(C)$, $\mathcal{M}_0^*(C) = \mathcal{M}^*(C)$, $\mathcal{B}_1(C) = \mathcal{B}(C)$ and $\mathcal{D}_0(C) = \mathcal{D}(C)$. Note that F_0 replaces F^* here and that the subscript for $\mathcal{B}$ is 1 rather than 0.

By induction, for each $t \in \omega$ we define a family $\mathcal{B}_t(C)$ of sets so that, for each $B \in \mathcal{B}_t(C)$, B^* is a separable, compact, subset of C and $\langle \bigcup \mathcal{L}_B, \leqslant_B \rangle$ is a maximal, n-flat, ordered set from $\mathcal{K}_n(B^*)$. We then let $\mathcal{M}(B)$, $\mathcal{M}^*(B)$, $\mathcal{B}(B)$ and $\mathcal{D}(B)$ be a type 8 construction for $\langle F_t, n, B^*, \langle \bigcup \mathcal{L}_B, \leqslant_B \rangle \rangle$ for each $B \in \mathcal{B}_t(C)$. Next define

$$\mathcal{M}_t(C) = \bigcup \{\mathcal{M}(B) \mid B \in \mathcal{B}_t(C)\}, \qquad \mathcal{M}_t^*(C) = \bigcup \{\mathcal{M}^*(B) \mid B \in \mathcal{B}_t(C)\},$$
$$\mathcal{B}_{t+1}(C) = \bigcup \{\mathcal{B}(B) \mid B \in \mathcal{B}_t(C)\}, \qquad \mathcal{D}_t(C) = \bigcup \{\mathcal{D}(B) \mid B \in \mathcal{B}_t(C)\}.$$

Recall that in defining the "type 8" construction we reindexed $\bigcup \mathcal{L}$ so that $L_0 < L_1$ for every $L \in \mathcal{L}$ in $\langle \bigcup \mathcal{L}, \leqslant \rangle$ and we then reindexed all of $\mathcal{K}_n(C)$ so that for $K \in (\mathcal{K}_n(C) - \mathcal{L})$, $\bigcup \mathcal{L} \subset V_0(K)$. Observe that we do not need to do any further reindexing once it it has been done for C at the zero level of this "type 11" construction. By induction, if this order is correct for some $B_t \in \mathcal{B}_t(C)$, then for each $B \in \mathcal{B}(B_t)$, we chose $\langle \mathcal{L}_B, \leqslant_B \rangle$ so that $L_0 <_{B_t} L_1$ for every $L \in \mathcal{L}_B$ and, for all $K \in (\mathcal{K}_n(B) - \mathcal{L}_B)$, $V_0(K_0) \supset \mathcal{L}_B$ because $V_0(K) \supset \mathcal{L}_{B_t}$.

Also observe that the members of $\bigcup_{t\in\omega} \mathcal{D}_t(C)$ are disjoint. One sees this by induction too. The members of $\mathcal{D}_0(C)$ are trivially disjoint. There may be $B \neq B'$ in $\mathcal{B}(C)$ with $B^* \cap (B')^* \neq \emptyset$ only because $q_B = q_{B'}$. But no matter whether $\{q_B\} = (\overline{B} - B)$ or whether q_B is the first term of $\langle \bigcup \mathcal{L}_B, \leqslant_B \rangle$, the first term of $\mathcal{M}(B)$ is $\{q_B\}$ and this term is not in $\mathcal{M}^*(B)$. Since every $D \in \mathcal{D}(B)$ is a subset of some $M \in \mathcal{M}^*(B)$, the terms of $\mathcal{D}_1(C)$ are all disjoint and disjoint from the terms of $\mathcal{D}_0(C)$, and the same argument repeats inductively for all $t \in \omega$.

It remains to define $\mathcal{D}_\omega(C)$. Let

$$\mathcal{D}_\omega^* = \left\{ \bigcap_{t\in\omega} M_t \mid M_t \in \mathcal{M}_t(C) \text{ and } M_{t+1} \subset M_t \right\}.$$

If $D = \left(\bigcap_{t\in\omega} M_t\right) \in \mathcal{D}_\omega^*$, then, for all $t \in \omega$, there is $B_t \in \mathcal{B}_t(C)$ with $M_{t+1} \in \mathcal{M}(B_t)$ and some $L_t \in \mathcal{L}_{B_t}$ such that $(L_t)_0 <_{B_t} (L_t)_1$ and $M_{t+1} \subset V_1(L_t)$. Thus $\{L_t \mid t \in \omega\}$ with the induced $(L_t)_0 < (L_t)_1$ order on its union is an n-flat ordered set. By Lemma 6, if $A = \bigcup\{V_1(L_t) \mid t \in \omega\}$, since the t have no maximum, there is a unique $q_D \in D$ in $(\overline{A} - A)$. Observe that $C \subset (D \cup A)$. So D is compact and, since $(D - \{q_D\})$ is open in the separable set C, D is separable. Define

$$\mathcal{D}_\omega(C) = \{D \in \mathcal{D}_\omega^* \mid D \neq \{q_D\}\};$$

$\mathcal{D}_\omega(C)$ is countable and its members are disjoint and do not meet any term of $\bigcup_{t\in\omega} \mathcal{D}_t(C)$.

Lemma 12. *If $D \in \bigcup_{t\leqslant\omega} \mathcal{D}_t(C)$, $|\mathcal{K}_n(D)| \leqslant 1$.*

Proof. Suppose $D \in \mathcal{D}_t(C)$ for some finite t. For some $B \in \mathcal{B}_t(C)$ and $M \in \mathcal{M}^*(B)$, $D = D_M$. For every $\alpha < 2^{|X|}$ for which it was possible we chose $\mathcal{L}_\alpha \subset \left(\mathcal{K}_n(M) - \bigcup_{\beta<\alpha} V_\beta\right)$ and $V_\alpha \subset (X - D)$. Thus there is no term of $\mathcal{K}_n(M)$ contained in D; but $D \subset M$ so $\mathcal{K}_n(D) = \emptyset$.

Suppose $D \in \mathcal{D}_\omega(C)$, $K \in \mathcal{K}_n(D)$ and $K \subset (D - \{q_D\})$ which is open in C. There is $t \in \omega$ with $K \subset (F_t \cap C)$. There is a unique $B \in \mathcal{B}_{t+1}(C)$ with $D \subset B$ and there are $B' \in \mathcal{B}_t(C)$ and $M \in \mathcal{M}^*(B')$ with $B' \in \mathcal{B}_t(C)$ and $B \in \mathcal{B}_M$. In the inductive process by which $\mathcal{L}_B$ was chosen, $\mathcal{L}_B = \mathcal{L}_\alpha$ for some α and $D \subset B \subset V_\alpha$ which is disjoint from V_β for all $\beta < \alpha$. Since $K \in \left(\mathcal{K}_n(M) - \bigcup_{\beta<\alpha} V_\beta\right) \cap F_t$, some $L \in (\mathcal{K}_n(D) \cap F_t)$ is in $\mathcal{L}_B$. But no $L \in \mathcal{L}_B$ is contained in D so we have a contradiction and if $\mathcal{K}_n(D) \neq \emptyset$, q_D is in the unique term of $\mathcal{K}_n(D)$.

8. The final construction

Construction 13. For all $n \in \omega$ we define $\mathcal{C}_n$, $\mathcal{D}_n$, $\mathcal{G}_n$ and P_n as well as Y and f.

For $x \in X$ and $t \in \omega$ let $F_t(x)$ be the term of $\mathcal{F}_t$ to which x belongs. Let $U_t(x) = F_t(x)$ if $x \notin \mathcal{K}'$ and $U_t(x) = F_t(x) \cap V_i(K)$ if $x = K_i \in K \in \mathcal{K}'$. Choose a one-to-one correspondence $\sigma : \omega \to \omega^2$ in such a way that $\sigma(n) = \langle s, r\rangle$ implies $s \leqslant n$. Let $C_n(x)$ and $D_n(x)$ be the terms of $\mathcal{C}_n$ and $\mathcal{D}_n$ to which x belongs (if any).

By induction on $n \in \omega$, for each $n \in \omega$ we define countable families $\mathcal{G}_n$, $\mathcal{C}_n$ and $\mathcal{D}_n$ of compact, separable, disjoint subsets of X and, for each $r \in \omega$, we choose $p_{nr} \in X$ as follows.

Let $\mathcal{G}_0 = \{X\}$. Suppose $n \in \omega$ and $\mathcal{G}_m$ has been chosen for all $m \leqslant n$ in such a way that $0 \leqslant m < n$ implies each $G \in \mathcal{G}_n$ is contained in a term of $\mathcal{D}_m$. Define

$$\mathcal{C}_n = \{G \cap F \mid G \in \mathcal{G}_n,\ F \in \mathcal{F}_n\}.$$

Recall that $\mathcal{F}_0 = \{X\}$ so $\mathcal{C}_0 = \{X\}$.

For each $C \in \mathcal{C}_n$ choose a maximal n-flat ordered set $\langle \bigcup \mathcal{L}_C, \leqslant_C\rangle$ from $\mathcal{K}_n(C)$ with $\mathcal{L}_C \cap \mathcal{K}_{n-1}(C) \neq \emptyset$ if $\mathcal{K}_{n-1}(C) \neq \emptyset$, and proceed to make a "type 11" construction: $\mathcal{M}_t(C)$, $\mathcal{M}_t^*(C)$, $\mathcal{B}_t(C)$ for $t < \omega$ and $\mathcal{D}_t(C)$ for $t \leqslant \omega$ from $\langle n, C, \langle \mathcal{L}_C \leqslant_C\rangle\rangle$.

Then define $\mathcal{M}_n = \bigcup\{\mathcal{M}_t(C) \mid t \in \omega,\ C \in \mathcal{C}_n\}$, $\mathcal{M}_n^*$, and $\mathcal{B}_n$ similarly, and $\mathcal{D}_n = \bigcup\{\mathcal{D}_t(C) \mid t \leqslant \omega,\ C \in \mathcal{C}_n\}$.

Suppose $x \in X$ and $m \leqslant n$. Let

$$\Delta_m(x) = \Bigl\{d \in \bigcup \mathcal{D}_m \mid \text{ either } d = x \text{ or } d = q_B \text{ for some } B \in \mathcal{B}_m \text{ with } x \in B\Bigr\}.$$

Observe that since $\Delta_m(x) \subset \bigcup_{t \leqslant \omega} \mathcal{D}_t(C_m(x))$ we can see that $\{D_m(d) \mid d \in \Delta_m(x)\}$ are distinct (and disjoint).

Assume that $X \neq \emptyset$, choose $\{p_{nr} \mid r \in \omega\}$ from X containing $\bigcup\{Q_M \mid M \in \mathcal{M}_n^*\}$ which is countable. Suppose $\sigma(n) = \langle s, r\rangle$, so $s \leqslant n$ and $p_{sr} = p$ has been defined. For $s \leqslant m \leqslant n$ we define $P_m(p)$ by induction as follows. Let $P_s(p) = \Delta_s(p)$ and for $s \leqslant m < n$ define $P_{m+1}(p) = \bigcup\{\Delta_m(x) \mid x \in P_m(p)\}$. Define $P_n = P_n(p)$. By induction, since $\{D_m(d) \mid d \in P_m(p)\}$ are distinct, $\{D_n(x) \mid x \in P_n\}$ are distinct.

If $D \in \mathcal{D}_n$ let $\mathcal{G}_{n+1}(D) = \{D\}$ if $D \neq D_n(x)$ for any $x \in P_n$, and if $x \in P_n$ define

$$\mathcal{G}_{n+1}\bigl(D_n(x)\bigr) = \bigl\{D_n(x) \cap \bigl(U_t(x) - U_{t+1}(x)\bigr) \mid t \in \omega\bigr\}.$$

We say

$$\bigl(D_n(x) \cap \bigl(U_t(x) - U_{t+1}(x)\bigr)\bigr) \leqslant_D \bigl(D_n(x) \cap \bigl(U_{t'}(x) - U_{t'+1}(x)\bigr)\bigr)$$

if $t \leqslant t'$. Define $\mathcal{G}_{n+1} = \bigcup\{\mathcal{G}_{n+1}(D) \mid D \in \mathcal{D}_n\}$.

Definition of Y and f

Suppose $x \in X$. If $n \in \omega$ and $M \in \mathcal{M}_n^*$ choose copies x_{n0} of x if $x = a_M$ and x_{n1} of x if $x = z_M$ ($x_{n0} \neq x_{n1}$ even if $x = a_M = z_M$). If $x = a_M$ or z_M, $D_M = D_n(x)$, hence there is at most one $M \in \mathcal{M}_n^*$ that might cause an x_{n0} or x_{n1} to be chosen and $x \notin D \in \mathcal{D}_\omega(C_n(x))$.

If $n \in \omega$ and $x = q_D$ where $D = D_n(x) \in \mathcal{D}_\omega(C_n(x))$, also choose copies x_{n0} and x_{n1} of x. Again x_{n0} and x_{n1} are unique if defined and this time both or neither is defined.

If $n \in \omega$ and $M \in \mathcal{M}_n^*$, give a one-to-one listing of $\mathcal{B}_M$ as $\{B_r(M) \mid r \in R_M\}$ where $R_M \subset \{\text{even } r \in \omega \mid r \geqslant 2\}$. If $M \in \mathcal{M}_n^*$, let $R_n(x) = \{r \in R_M \mid x = q_{B_r(M)}\}$ and, for all $r \in R_n(x)$, let x_{nr} and $x_{n(r+1)}$ be copies of x.

For $n \in \omega$ let $E_n(x) = \{x\} \cup \{\text{defined } x_{nr} \mid r \in \omega\}$ and $E(x) = \bigcup_{n\in\omega} E_n(x)$. Let $E_n(x)^+ = \bigcup\{E_m(x) \mid m > n\}$. Define $Y = \bigcup\{E(x) \mid x \in X\}$ and define $f : Y \to X$ by $f(y) = x$ if $y \in E(x)$.

9. Some partial orders on Y determining $\leqslant$

We use y, y', y'' for members of Y. Recall that $\mathcal{G}_0 = \mathcal{C}_0 = \{X\}$. We define an initial partial order $\leqslant$ on Y as follows.

Suppose $t \in \omega$ and $B \in \mathcal{B}_t(X)$. If $M \in (\mathcal{M}(B) - \mathcal{M}^*(B))$ and $a_M \neq z_M$ are both defined, define $a_M < z_M$. If $f(y) \in M \in \mathcal{M}(B)$, $f(y') \in M' \in \mathcal{M}(B)$, and $M < M'$ in $\langle \mathcal{M}(B), \leqslant\rangle$, define $y < y'$ unless $t > 0$ and $f(y) = q_B$.

Suppose $t \in \omega$, $M \in \mathcal{M}_t^*(X)$, $B = B_r(M)$, and $x = q_B$. Define $x_{0r} < y < x_{0(r+1)} < x$ whenever $f(y) \in B$ and, if $B' = B_{r'}(M)$, $x = q_{B'}$, $f(y') \in B'$, and $r < r'$ in $R_0(x)$, define $y < x_{0(r+1)} < x_{0r'} < y'$.

If $M \in \mathcal{M}_t^*(X)$ and $x = a_M$ or $x' = z_M$, define $x_{00} \leqslant y$ and $y \leqslant x'_{01}$ whenever $f(y) \in M$.

If $x = q_D$ for some $D \in \mathcal{D}_\omega(X)$, then define $x_{00} \leqslant y \leqslant x_{01}$ whenever $f(y) \in D$.

It is easy to check that $\leqslant$ as defined above is transitive and thus $\leqslant$ is a partial order on Y. Let $\leqslant_0$ denote this partial order. By induction, for each $n \in \omega$ we define a partial order $\leqslant_n$ on Y with $\leqslant_{n+1}$ extending $\leqslant_n$, a process that we prove in Section 10 converges to a total order $\leqslant$ on Y.

Suppose the partial order $\leqslant_n$ on Y has been defined. If $x \in X$, define

$$Y_n(x) = \{x\} \cup \{y \in Y \mid x_{n'r} \leqslant_n y \leqslant_n x_{n'(r+1)} \text{ for some } n' \leqslant n \text{ and } r \in R_{n'}(x)\}.$$

Let $Y_n(x)^+ = (E_n(x)^+ \cup Y_n(x))$. If D is in $\mathcal{D}_n$ or $\mathcal{G}_{n+1}$ or $\mathcal{C}_{n+1}$, let $D^+ = \bigcup\{Y_n(x)^+ \mid x \in D\}$.

We inductively assume:

(1) $Y_n(x)$ is an interval of $\leqslant_n$ with x as its last term and $x_{n'r'} <_n x_{n''r''}$ whenever $n' < n''$ or $n' = n''$ and $r' < r''$.

(2) If $y \in E_n(x)^+$, y is not $<_n$ related to any term of $Y_n(x)^+$, but $x <_n y'$ for some $y \notin Y_n(x)^+$ if and only if $y <_n y'$ (and the same with $>_n$ replacing $<_n$).

(3) If $D \in \mathcal{D}_n$, $\{Y_n(x) \mid x \in D\}$ are disjoint and, if $x \in D$, $x' \in D$, and $y \notin (Y_n(x) \cup Y_n(x'))$, then $x <_n y$ if and only if $x' <_n y$ (and similarly with $>_n$ replacing $<_n$).

(4) If $x \in D \in \mathcal{D}_n$, $x' \in D' \in \mathcal{D}_n$, $D \neq D'$, and $Y_n(x) \cap Y_n(x') \neq \emptyset$, then $D^+ \subset Y_n(x')$ or $(D')^+ \subset Y_n(x)$.

It is easy to check that these four conditions hold for $n = 0$.

We begin to extend $\leqslant_n$ to $\leqslant_{n+1}$ by requiring that if $D = D_n(x)$ for some $x \in P_n$ and $G <_D G'$ in $\langle \mathcal{G}_{n+1}(D), \leqslant_D\rangle$, then $y'' <_{(n+1)} y' <_{(n+1)} y$ whenever $y'' \in Y_n(x)$,

$y' \in (G')^+$ and $y \in G^+$. (And the reduced formula gotten by deleting y' and G' also holds.) Total order $\mathcal{F}_{n+1}$ as $\langle \mathcal{F}_{n+1}, \leqslant \rangle$. Again extend $\leqslant_n$ by requiring that $G \in \mathcal{G}_{n+1}$ and $F < F'$ in $\langle \mathcal{F}_{n+1}, \leqslant \rangle$ yields $y <_{(n+1)} y'$ for $y \in (F \cap G)^+$ and $y' \in (F' \cap G)^+$.

If $C \in \mathcal{C}_{n+1}$, then $C = F \cap G$ for some $F \in \mathcal{F}_{n+1}$ and $G \in \mathcal{G}_{n+1}$. Since our extensions of $\leqslant_n$ did not disturb any $Y_n(x)$ and only took disjoint refinements of members of $\mathcal{D}_n$, we have preserved transitivity as well as our four properties with (3) and (4) replaced by:

(3*) If $C \in \mathcal{C}_{n+1}$, $\{Y_n(x) \mid x \in C\}$ are disjoint and, if $x \in C$, $x' \in C$, and $y \notin (Y_n(x) \cup Y_n(x'))$, then $x <_n y$ if and only if $x' <_n y$ (and similarly with $>_n$ replacing $<_n$).

(4*) If $x \in C \in \mathcal{C}_{n+1}$, $x' \in C' \in \mathcal{C}_{n+1}$, $C \neq C'$, and $Y_n(x) \cap Y_n(x') = \emptyset$, then either $C^+ \subset Y_n(x')$ or $(C')^+ \subset Y_n(x)$.

We continue to construct $\leqslant_{n+1}$ from $\leqslant_n$ using $\mathcal{C}_{n+1}$.

Suppose $C \in \mathcal{C}_{n+1}$; recall that

$$C^+ = \bigcup \{Y_n(x) \cup E_n(x)^+ \mid x \in C\}.$$

For the purposes of the following construction we assume $y, y', y'', \ldots$ are terms of C^+. We define a partial order $\leqslant$ on C^+. But first define $f_C : C^+ \to X$ by $f_C(y) = x$ if $y \in Y_n(x)$ and $f_C(y) = f(y)$ if $y \notin Y_n(x)$ for any $x \in C$. Since $\{Y_n(x) \mid x \in C\}$ are disjoint, f_C is well defined.

Suppose $t \in \omega$ and $B \in \mathcal{B}_t(C)$. If $M \in (\mathcal{M}(B) - M^*(B))$ and $a_M \neq z_M$ are both defined, define $y < y'$ if $f_C(y) = a_M$ and $f_C(y') = z_M$. If $f_C(y) \in M \in \mathcal{M}(B)$, $f_C(y') \in M' \in \mathcal{M}(B)$, and $M < M'$ in $\langle \mathcal{M}(B), \leqslant \rangle$, define $y < y'$ unless $t > 0$ and $f_C(y) = q_B$.

Suppose $t \in \omega$, $M \in \mathcal{M}_t^*(C)$, $B = B_r(M)$ and $x = q_B$. Define $y'' < x_{(n+1)r} < y < x_{(n+1)(r+1)} < x$ whenever $f_C(y) \in B$ and $y'' \in (Y_n(x) - \{x\})$. (Delete y'' if $(Y_n(x) - \{x\}) = \emptyset$.) If $B' = B_{r'}(M)$, $x = q_B$, $f_C(y') \in B'$, and $r < r'$ in $R_{(n+1)}(x)$, define $y < x_{(n+1)(r+1)} < x_{(n+1)r'} < y'$.

If $M \in \mathcal{M}_t^*(C)$ and $x = a_M$ or $x' = z_M$, define $x_{(n+1)0} \leqslant y$ and $y \leqslant x'_{(n+1)1}$ whenever $f_C(y) \in M$.

If $x = q_D$ for some $D \in \mathcal{D}_\omega(C)$, then define $x_{(n+1)0} \leqslant y \leqslant x_{(n+1)1}$ whenever $f_C(y) \in D$.

By (3*) and (4*), for the same reasons that $\leqslant_0$ was a partial order on Y, this $\leqslant$ on C^+ is also a partial order. Define $y \leqslant_{n+1} y'$ in Y if and only if $y \leqslant_n y'$ or y and y' are both in C^+ for some $C \in \mathcal{C}_{n+1}$ and $y \leqslant y'$ in the order just defined for C^+. Since the terms in a particular $Y_n(x)$ are contained in $Y_{n+1}(x)$, (1)–(4) are again satisfied by $\leqslant_{n+1}$.

Let $\leqslant$ be the minimal order on Y extending $\leqslant_n$ for all $n \in \omega$; $\leqslant$ is a (transitive) partial order on Y.

10. Totality of the order

Lemma 14. *$\leqslant$ is a total order on Y.*

Proof. Suppose y and y' are in Y and unrelated by $\leqslant$.

It is easy to see that $f(y) \neq f(y')$. If there is a minimal n for y or y' being x_{n0} or x_{n1}, say y is x_{n0}. Then $y <_n y'$. And if $y = x_{n1}$, $y >_n y'$. Otherwise, if $f(y) = f(y') = x$, there is an n for which both y and y' are in $Y_n(x)$ and y and y' are ordered by $\leqslant_n$ in accordance with (1) of the induction hypothesis.

Since y and y' are unrelated by $\leqslant_0$, one can read through the construction of $\leqslant_0$ and discover that there is some $D \in \mathcal{D}_0$ such that both y and y' are in D^+ and either

(1) both $f(y)$ and $f(y')$ are in D, or

(2) $D = D_M$ for some $M \in \mathcal{M}_0^*$ and there is $B \in \mathcal{B}_M$ such that $f(y) \in B$ and $f(y') \in (D - \{q_B\})$ (or vice versa), or

(3) $f(y) \in B \in \mathcal{B}_M$ and $f(y') \in B' \in \mathcal{B}_M$ while $q_B \neq q_{B'}$.

If $f(y) \in B$, $y \in Y_0(q_B)$ and, if $f(y') \in B'$, $y' \in Y_0(q_{B'})$.

Suppose that for all $n \in \omega$ there is $D_n \in \mathcal{D}_n$ such that $f(y)$ and $f(y')$ are both in D_n. By Lemma 12, if $n \in \omega$, $|\mathcal{K}_n(D_n)| \leqslant 1$. There is $C \in \mathcal{C}_{n+1}$ with $D_{n+1} \subset C$ so $C \subset D_n$. Since $\mathcal{K}_n(C) \subset \mathcal{K}_n(D_n)$, $|\mathcal{K}_n(C)| \leqslant 1$. If $\mathcal{K}_n(C) \neq \emptyset$ we chose $\mathcal{L}_C$ in such a way that there is a term of $\mathcal{K}_n(C)$ in $\mathcal{L}_C$. Since no term of $\mathcal{L}_C$ is in any $M \in \mathcal{M}_0(C)$ and, if $\mathcal{K}_n(C) \neq \emptyset$, there is such an M containing D_{n+1}, $\mathcal{K}_n(D_{n+1}) = \emptyset$. Each D_{n+1} is contained in a term of $\mathcal{F}_n$ and $\bigcap_{n\in\omega} D_n \notin \mathcal{K}'$. So $\bigcap_{n\in\omega} D_n$ is a single point of X and cannot be both $f(y)$ and $f(y')$ which are different. Thus there is a minimal $s \in \omega$ such that $f(y)$ and $f(y')$ are not both in the same term of $\mathcal{D}_s$.

If $n < s$, there is $D_n \in \mathcal{D}_n$ such that both $f(y)$ and $f(y')$ are in D_n. One should observe that $y \in Y_n(f(y))^+$ and $y' \in Y_n(f(y'))^+$. If this fails for some minimal $n < s$, (say) $y \notin Y_n(f(y))^+$, then y is x_{n0} or x_{n1} where $f(y) = x$. There is $C \in \mathcal{C}_n$ with $D_n \subset C$. If $x = a_M$ or z_M for some $M \in \mathcal{M}_t(C)$ with $t \in \omega$, then $D_n = D_M$. If $y = x_{n0}$, $x = a_M$ and $y < y'$. If $y = x_{n1}$, $x = z_M$ and $y' < y$. Otherwise $D \in \mathcal{D}_\omega(C)$ and $x = q_D$; again $y = x_{n0}$ implies $y < y'$ and $y = x_{n1}$ implies $y' < y$. So, if $s > 0$, there is $D \in \mathcal{D}_{s-1}$ such that both $f(y)$ and $f(y')$ are in D, $y \in Y_{s-1}(f(y))^+$, and $y' \in Y_{s-1}(f(y'))^+$.

Since y and y' are $\leqslant q_s$ unrelated, if $s \geqslant 0$, there is $C \in \mathcal{C}_s$ such that $f(y)$ and $f(y')$ are both in C. ($C = X$ if $x = 0$.) As in the case $s = 0$, checking through the definition of $\leqslant q_s$, there is $D \in \mathcal{D}_s$ with both y and y' in D^+ and there are $t \in \omega$, $M \in \mathcal{M}_t(C)$ with $D = D_M$ and $B \in \mathcal{B}_M$ such that one of $f(y)$ and $f(y')$, say $f(y) \in B$, and the other, $f(y')$, is either in $B' \in \mathcal{B}_M$ with $q_B \neq q_{B'}$, or $f(y') \in D$ and $f(y') \neq q_B$. Let $D_s = D$ and $y_s = q_B$; $y \in Y_s(y_s)$. If $f(y') \in B'$, let $y'_s = q_{B'}$; $y' \in Y_s(y'_s)$. If $f(y') \in D$, $y' \in Y_s(f(y'))^+$; let $y' = y'_s$ in this case.

Observe that:

(1) y and y' are $\leqslant q$ ordered if y_s and y'_s are

(2) $y_s \in D_s \cap Q_M$ for some $M \in \mathcal{M}_s$.

(3) Either $y'_s \in D_s$ or $y'_s = y'$.

(4) If $y'_s = y'$, $f(y') \in D_s$ and $y' \in Y_s(f(y'))^+$.

For all $s < n < \omega$ we inductively find $D_n \in \mathcal{D}_n$ and y_n and y'_n in D_n^+ such that:

(1) y_{n-1} and y'_{n-1} are $\leqslant$ ordered if y_n and y'_n are.

(2*) $y_n \in D_n \cap \Delta_n(y_{n-1})$.

(3) Either $y'_n \in D_n$ or $y'_n = y'$.

(4) If $y'_n = y', f(y') \in D_n$ and $y' \in Y_n(f(y'))^+$.

Since $y_s \in Q_M$, $y_s = p_{sr}$ for some $r \in \omega$ and there is $n \geqslant s$ such that $\sigma(n) = \langle s, r\rangle$. By (2*), $y_n \in P_n$. Since $\mathcal{G}_{n+1}(D_n) = \{D_n \cap U_t(y_n) \mid t \in \omega\}$ and by our definition of $\leqslant_{n+1}$, $y_n \leqslant_n y'_n$ contrary to (1).

Thus $\leqslant$ is a total order on Y. □

Topologize Y by giving it the $\leqslant$ order topology.

11. Compactness of Y

Suppose $x \in X$. Let us look at $E(x)$ *as a subspace of* Y.

$$E(x)^* = \{x\} \cup \{x_{nr} \mid n \in \omega,\ r > 2\}$$

under $\leqslant$ has x as its last point, any term of Y between two members of $E(x)^*$ is between x_{nr} and $x_{n(r+1)}$ for some n and r, and the x_{nr}s are lexicographically ordered (see (1) of the induction hypotheses). However $\{x_{n0} \mid n \in \omega\}$ is well ordered by n in $\langle Y, \leqslant\rangle$ while $\{x_{n1} \mid n \in \omega\}$ is reverse well ordered and $E(x)^*$ is between them. If $E(x)^* \neq \{x\}$, then $x \in P(n)$ for some $n \in \omega$ and $E_n(x)^+ = \{x\}$. If $\{x_{n0} \mid n \in \omega\}$ is infinite (so $E(x)^* = \{x\}$), x is the least upper bound of $\{x_{n0} \mid n \in \omega\}$, and, if $\{x_{n1} \mid n \in \omega\}$ is infinite, x is the greatest lower bound of $\{x_{n1} \mid n \in \omega\}$. One consequence of all of this is that $E(x)$ is a compact subset of Y since $\langle E(x), \leqslant\rangle$ is Dedekind complete.

Lemma 15. *Y is compact.*

Proof. Suppose $I = \{y_\alpha \mid \alpha < \kappa\}$ is strictly increasing in $\langle Y, \leqslant\rangle$ for some ordinal κ of infinite cofinality, and assume there is no minimal element of Y greater than all of the terms of I. We show this is impossible and a similar argument proves every decreasing sequence has a lower bound, thus proving that $\langle Y, \leqslant\rangle$ is Dedekind complete and Y is compact.

If $\alpha < \kappa$, let $\alpha^+ = \{y_\beta \mid \alpha \leqslant \beta < \kappa\}$. For all $n \in \omega$, choose $C_n \in \mathcal{C}_n$, $D_n \in \mathcal{D}_t(C_n)$ for some $t \leqslant \omega$, and $\delta_n < \kappa$ such that $\delta_n^+ \subset D_n^+$ but $\sigma^+ \not\subset Y_n(x)^+$ for any $\sigma < \kappa$ and $x \in D_n$. Choose so $D_{n+1} \subset D_n$ as follows.

Let $C_0 = X$. One can think of $Y_{-1}(x)$ as $\{x\}$, so $Y_{-1}(x)^+ = E(x)$, so $\sigma^+ \not\subset Y_{-1}(x)^+$ for any σ. Trivially, if $\gamma_0 = 0, \gamma_0^+ \subset C_0^+$. If $n > 0$ and D_{n-1} and δ_{n-1} have been chosen, there are $C_n \in \mathcal{C}_n$ with $C_n \subset D_{n-1}$ and $\delta_{n-1} \leqslant \gamma_n < \kappa$ such that $\gamma_n^+ \subset C_n$ (but $\sigma^+ \not\subset Y_{n-1}(x)^+$ for any $\sigma < \kappa$ and $x \in C_n \subset D_{n-1}$).

Having chosen C_n, for each $t \in \omega$ for which it is possible, choose $B_t \in \mathcal{B}_t(C_n)$ and $\gamma_n \leqslant \beta_t < \kappa$ such that $\beta_t^+ \subset B_t^+$. Since $\mathcal{B}_0(C_n) = \{C_n\}$, $B_0 = C_n$ and $\beta_0 = \gamma_n$ have the desired property. Thus either B_t and β_t can be defined for all $t \in \omega$ or:

Case (a). *There is a maximal $t \in \omega$ for which B_t and β_t can be defined.*

Recall that every $Y_n(x)$ for $x \in X$ has a first and last term in $\langle Y, \leqslant\rangle$; let $J(x)$ be the set of these. Also if $x \in B_t \cap (\bigcup(\mathcal{M}(B_t) - \mathcal{M}^*(B_t)))$, $Y_n(x)^+ = Y_n(x) = Y_{n-1}(x)$.

Let

$$\begin{aligned} J = \bigcup \Big\{ J(x) \mid x \in B_t \cap \Big(\bigcup \big(\mathcal{M}(B_t) - \mathcal{M}^*(B_t)\big)\Big)\Big\} \\ \cup \big\{ x_{n0} \mid x = a_M \text{ for some } M \in \mathcal{M}^*(B_t)\big\} \\ \cup \big\{ x_{n1} \mid x = z_M \text{ for some } M \in \mathcal{M}^*(B_t)\big\} \\ \cup \big\{ x_{nr} \mid t > 0,\ x = q_{B_t},\ B_t = B_{rM} \text{ for some } M \in \mathcal{M}_{t-1}(C_n)\big\}. \end{aligned}$$

Since J is Dedekind complete in $\langle Y, \leqslant\rangle$, there are $\beta_t < \delta_n < \kappa$ and $M \in \mathcal{M}^*_{t-1}(C_n)$ such that $\delta_n^+ \subset M^+$. Let $D_n = D_M$. Since $M = D_M \cup (\bigcup \mathcal{B}_M)$ and for all $\mathcal{B} \in \mathcal{B}_M$ where $B = B_r(M)$ and $x = q_B$, $x_{nr} < B^+ < x_{n(r+1)}$ and $B^+ \subset Y_n(x)$, $M^+ = D_M^+$. Thus $\delta_n^+ \subset D_n^+$.

If $B \in \mathcal{B}_M$, B^+ is an interval of $\leqslant$. Hence, since we are in Case (a),

$$\tau_B = \sup\big\{\alpha < \kappa \mid y_\alpha \in B^+\big\} < \kappa.$$

If $x \in D_n$, $(Y_n(x)^+ - Y_{n-1}(x)^+) \subset \bigcup \mathcal{B}_n(x)^+$ where $\mathcal{B}_n(x) = \{B^+ \mid B \in \mathcal{B}(M)$ and $x = q_B\}$. Let $N(x) = \{m > n \mid R_m(x) \neq \emptyset\}$. If $\mathcal{B}_n(x)$ is infinite, the minimal term of Y greater than $\bigcup \mathcal{B}_n(x)$ is x if $N(x)$ is $\emptyset$ and is x_{mr} where m is minimal in $N(x)$ and r is minimal in $R_m(x)$ otherwise. Thus, for all $x \in D_n$, there is no $\sigma < \kappa$ with $\sigma^+ \subset Y_n(x)^+$.

In Case (a) we can thus choose appropriate C_n, D_n and δ_n.

Case (b). *For all $t \in \omega$ we can choose $B_t \in \mathcal{B}_t(C_n)$ and $\beta_t < \kappa$ such that $\beta_t^+ \subset B_t^+$.*

Since $B_{t+1} \subset B_t$, there is a $D_n = (\bigcap_{t\in\omega} B_t) \in \mathcal{D}^*_\omega(C_n)$. If $t > 0$, $M_t \in \mathcal{M}_t(C_n)$, $B_t = B_r(M_t)$, and $x = q_{B_t}$, let $u_t = x_{nr}$ and $u'_t = x_{n(r+1)}$. Then $u_t < B_t^+ < u'_t$ in $\langle Y, \leqslant\rangle$. Suppose $v = q_D$. If $D = \{v\}$, then

$$Y_n(v) = \{y \in Y \mid u_t < y < u'_t \text{ for all } t \in \omega\}$$

and since $Y_n(v)$ is a compact subinterval of $\langle Y, \leqslant\rangle$, I has a least upper bound. If $D \neq \{v\}$, then $u_t < v_{n0} < D_n^+ < v_{n1} < u'_t$ for all $t > 0$, v_{n0} is the first point of Y greater than $\{u_t \mid t \in \omega\}$ and v_{n1} is the last point of Y less than $\{u'_t \mid t \in \omega\}$, and $u_t < \beta_t^+ < u'_t$. Thus, since $\{y_\alpha \mid \alpha < \kappa\}$ has no upper bound, there is $\delta_n < \kappa$ such that $\delta_n^+ \subset D_n^+$.

If $x \in D_n$, $Y_n(x)^+ = Y_{n-1}(x)^+$ unless $x = v$ in which case $Y_n(v)^+ = Y_{n-1}(v)^+ - \{v_{n0}, v_{n1}\}$. Thus, since $D_n \subset D_{n-1}$ if $n > 0$ (and $Y_{-1}(x) = \{x\}$ if $n = 0$), if $\sigma < \kappa$, $\sigma^+ \not\subset Y_n(x)$. Hence C_n, D_n and δ_n can be defined for all $n \in \omega$.

Case (c). *We prove this leads to a contradiction.*

As proved in Section 10, $\bigcap_{n\in\omega} D_n$ is a single point p.

If $n \in \omega$ choose an open U_n with $U_n \subset F_n$ and $p = \bigcap_{n\in\omega} U_n$. If $D_n \notin U$ for F and $\mathcal{D}_\omega(C_n)$ there is $M_n \in \mathcal{M}^*_n$ such that $D_n = D_{M_n}$. If $n \in \omega$, by Corollary 10,

$$\mathcal{B}(n) = \big\{B \in \mathcal{B}_m(M) \mid m \leqslant n,\ D_m = D_{M_n},\ q_B \subset D_n,\ \text{but } (B - U_n) \neq \emptyset\big\}$$

is finite. For each $x \in D_n$, there is $\tau_B < \kappa$ such that $\tau_B^+ \cap B^+ = \emptyset$. So there is $\delta_n < \tau_n < \kappa$ such that $\tau_n > \tau_B$ for all $B \in \mathcal{B}(n)$. If $x \in D_n$, then

$$Y_n(x)^+ \subset E(x) \cup \left\{ B^+ \mid B \in \bigcup_{m \leqslant n} \mathcal{B}_m(M), D_m = D_{M_m}, \text{ and } x = q_B \right\}.$$

Hence $f(D_n^+ \cap \tau_n^+) \subset U_n$. We can assume $\{\tau_n \mid n \in \omega\}$ is cofinal with κ. There is an infinite $N \subset \omega$ such that $n < n'$ in N implies $y_{\tau_n} \notin D_{n'}^+$. Since $p \in D_{n'}^+ \supset (\tau_{n'})^+$, and $D_{n'}^+$ is an interval of $\langle Y, \leqslant\rangle, y_{\tau_n} < p$ for all $n \in N$. So p is an upper bound on $\{y_\alpha \mid \alpha < \kappa\}$ in $\langle Y, \leqslant\rangle$, contrary to assumption. □

We have thus proved that Y is compact. The same proof yields:

Corollary 16. *Suppose $C \in \mathcal{C}_n$.*

(I) *C^+ is a compact subinterval of $\langle Y, \leqslant\rangle$ as are M^+ for $M \in \mathcal{M}_t(C)$ for some $t \in \omega$, D^+ for $D \in \mathcal{D}_t(C)$ some $t \leqslant \omega$, and G^+ for $G \in \mathcal{G}_{n+1}(D)$.*

(II) *Suppose $B \in \mathcal{B}_t(C)$, $t > 0$, $x = q_B$ and $B = B_r(M)$ for some $r \in R_n(x)$ and $M \in \mathcal{M}_t^*(C)$. Then the interval*

$$J(B) = [x_{nr}, x_{n(r+1)}] = \{x_{nr}\} \cup B^+ \cup \{x_{n(r+1)}\}$$

of $\langle Y, \leqslant\rangle$ has x_{nr} as its first point, $x_{n(r+1)}$ as its last, and is compact as are $[x_{nr}, x_{n(r+1)})$ and $(x_{nr}, x_{n(r+1)}) = B^+$ unless $q_B \in \overline{B}$.

(III) *If $D \in \mathcal{D}_\omega(C)$ and $x = q_D$, then the interval*

$$J(D) = [x_{n0}, x_{n1}] = \{x_{n0}\} \cup D^+ \cup \{x_{n1}\}$$

of $\langle Y, \leqslant\rangle$ has x_{n0} as its first point, x_{n1} as its last, and is compact as are $[x_{n0}, x_{n1}), (x_{n0}, x_{n1}]$, and $(x_{n0}, x_{n1}) = D^+$.

Proof. Assume that $\{y_\alpha \mid \alpha < \kappa\}$ is a strictly increasing (or decreasing) in $\langle Y, \leqslant\rangle$ subset of C^+. Almost exactly as in the proof of Lemma 15, using the facts that C is compact in X and C^+ is an interval of $\langle Y, \leqslant\rangle$, one proves that C^+ is compact in Y. The other proofs are analagous as are the proofs of (II) and (III). □

12. Continuity of f

To prove our theorem it remains to prove that f is continuous. To aid with this we begin with three "facts".

We make frequent use of Fact A. The members of $\mathcal{Z}$ may belong to $\mathcal{B}_t(C)$ for various t, or $\mathcal{M}_t(C)$ for some t, or $\mathcal{G}_{n+1}(D)$ for some $D \in \mathcal{D}_t(C)$. The plus operation on Z^* is in relation to the n for which $C \in \mathcal{C}_n$.

Fact A. *Suppose U is clopen in X, $n \in \omega$, $C \in \mathcal{C}_n$, and $\mathcal{Z}$ is a totally ordered unbounded family of disjoint subsets of C such that*

$$B \in \mathcal{B}_t(C) \text{ and } q_B \in Z \in \mathcal{Z} \text{ imply } B \subset Z^* = \bigcup\{Z' \in \mathcal{Z} \mid Z' \geqslant Z \text{ in } \mathcal{Z}\}. \quad (*)$$

If there is $Z_0 \in \mathcal{Z}$ such that $\overline{Z_0^*} \subset U$, then there is $Z \geqslant Z_0$ in $\mathcal{Z}$ such that $f((Z^*)^+) \subset U$.

Proof. Suppose $Z \geqslant Z_0$ in $\mathcal{Z}$, $y \in (Z^*)^+$ and $f(y) \notin U$. By $(*)$ there are $m < n$ and $C_m \in \mathcal{C}_m$ with $C_n \subset C_m$, $t \in \omega$ and $M \in \mathcal{M}_t(C_m)$ such that $f(y) \in B \in \mathcal{B}_M$ and $q_B \in Z_0^* \subset U$, but $(B - U) \neq \emptyset$. There are at most finitely many $m < n$. Since $C \subset D \in \mathcal{D}_m$, $C \subset C_m$ so M and t are determined by m. By Corollary 10 there are at most finitely many choices for B and thus for q_B. Since the members of $\mathcal{Z}$ are disjoint there is some $Z > Z_0$ in $\mathcal{Z}$ such that no q_B of the type described is in Z^*. Thus $f(Z^*)^+ \subset U$. $\square$

Comment. In one case (Case (4.3) of Lemma 17) we have $Z_n = (D_n - D_{n+1})$ where $\overline{Z_n^*} = D_n \in \mathcal{D}_n$ and $D_{n+1} \subset D_n$. Thus there is no fixed $n \in \omega$ or $C \in \mathcal{C}_n$ for $\mathcal{Z} = \{Z_n \mid n \in \omega\}$, nor does $(*)$ hold; and the plus operation on Z_n is relative to n which is different for different members of $\mathcal{Z}$. However $D_{n+1}^+ \subset D_n^+$ and the same argument yields an n with $f(D_n^+) \subset U$.

Fact B. *Suppose $C \in \mathcal{C}_n$. If $f(y) = x$ and y is the first point of C^+ in $\langle Y, \leqslant\rangle$, then y is either the first point of Y or y has an immediate predecessor in $\langle Y, \leqslant\rangle$. (Symmetrically, if y is the last point of C^+, y is either the last point of Y or y has an immediate successor.)*

Proof. We can assume that n is minimal for Fact B to fail for y and that $n > 0$. Let $D = D_{n-1}(x)$ and

$$\mathcal{C} = \{(C')^+ \mid C' \in \mathcal{C}_n,\ C' \subset D, \text{ and } (C')^+ \text{ precedes } C^+ \text{ in } \langle Y, \leqslant\rangle\}.$$

By our choice of $\mathcal{G}_n(D)$, $\leqslant_D$, and $\mathcal{C}_n$, either there is a maximal $(C')^+ \in \mathcal{C}$ in $\langle Y, \leqslant\rangle$ and the maximal term of $(C')^+$ is then the immediate predecessor of y in $\langle Y, \leqslant\rangle$, or y is the first point of D^+. If $D \in \mathcal{D}_\omega(C_{n-1}(x))$ and $q = q_D$, then $q_{(n-1)0}$ is the immediate predecessor of y. So we can assume y is the first point of D^+ and there are $M \in \mathcal{M}_t(C_{n-1}(x))$ and $B \in \mathcal{B}_t(C_{n-1}(x))$ for some $t \in \omega$ such that $M \in \mathcal{M}^*(B)$ and $D = D_M$. If a_M exists, $f(a_M)_{n0}$ is the immediate predecessor of y. Otherwise, if

$$\mathcal{M}' = \{M' \in \mathcal{M}(B) \mid M' < M \text{ in } \langle \mathcal{M}(B), \leqslant\rangle\},$$

either there is a maximal $M' \in \mathcal{M}'$ and, in this case, the last point of $(M')^+$ in $\langle Y, \leqslant\rangle$ is the immediate predecessor of y, or $\mathcal{M}' = \emptyset$. Thus we can assume $\mathcal{M}' = \emptyset$ and y is the first point of B^+. If $q = q_B$, $q_{(n-1)0}$ is then the immediate predecessor of y. If there is no q_B, $B = C_{n-1}(x)$. Then y is the first point of $C_{n-1}(x)^+$ contrary to our assumption on the minimality of n. $\square$

Fact C. *Suppose $y \in Y$, $f(y) = x \in U$ which is clopen in X, $m \in \omega$ is maximal for $C_m(x)$ to be defined, and y is minimal in $\{x\}^+ = Y_m(x)$. Then either y is minimal in Y or there is $a < y$ in $\langle Y, \leqslant\rangle$ with $f((a, y]) \subset U$. Symmetrically, if y is maximal in $\{x\}^+$, either y is maximal in Y or there is $b > y$ in $\langle Y, \leqslant\rangle$ with $f([y, b)) \subset U$.*

Proof. Either

(i) for all $t \in \omega$ there are $B_t \in \mathcal{B}_t(C_m(x))$ such that $\{x\} = \bigcap_{t\in\omega} B_t$, or

(ii) $x \in P_m$, or

(iii) there are $t \in \omega$, $B \in \mathcal{B}_t(C_m(x))$, and $M \in (\mathcal{M}(B) - \mathcal{M}^*(B))$ such that $x \in M$ and $x \neq q_B$.

Suppose y is minimal in $\{x\}^+$.

In case (i), by Fact A, since there is $t \in \omega$ with $\overline{B_t} \subset U$, there is $t' > t$ such that $f(B_{t'}^+ - \{x\}^+) \subset U$. Thus, if a is the first point of $J(B_{t'})$, since $\{x\}^+$ is a subinterval of $B_{t'}^+ \subset J(B_{t'})$, the interval $(a, y]$ of $\langle Y, \leqslant\rangle$ has $f((a, y]) \subset B_{t'}^+ \subset U$.

In case (ii), let $D = D_m(x)$. Since $x \in P_m$, $\{x\}^+$ is an initial interval of D^+ in $\langle Y, \leqslant\rangle$ and, by our assumption that y is minimal in $\{x\}^+$, y is the first point of D^+.

On the other hand, suppose that y is maximal in $\{x\}^+$. If there is a maximal G in $\langle \mathcal{G}_{m+1}(D), \leqslant_D\rangle$ and b is the first point of G^+, then b is the immediate successor of y. If $\mathcal{G}_{m+1}(D)$ is infinite, there is $G \in \mathcal{G}_{m+1}(D)$ with

$$G^* = \bigcup \{G' >_D G \text{ in } \langle \mathcal{G}_{m+1}(D), \leqslant_D\rangle\} \subset U.$$

By Fact A we can find $G' >_D G$ in $\mathcal{G}_{m+1}(D)$ with $f(((G')^*)^+) \subset U$. If b is the last point $(G')^+$, then $y < b$ in $\langle Y, \leqslant\rangle$ and $f([y, b)) \subset U$. Otherwise, $\mathcal{G}_{m+1}(D) = \emptyset$ and y is the maximal point of D^+.

We return to our assumption that y is minimal in $\{x\}^+$ and thus, in case (ii), in D^+. If $D \in \mathcal{D}_\omega(C_m(x))$, $q = q_D$, and $a = q_{m0}$, then a is the immediate predecessor of y. Suppose instead that there are $t \in \omega$, $B \in \mathcal{B}_t(C_m(x))$, $M \in \mathcal{M}^*(B)$, and $D = D_M$ with y being the first point of D^+. If $p = a_M$ and $a = p_{m0}$, a is the immediate predecessor of y. If there is no a_M, y is the first point of M^+. Since $M \in \mathcal{M}^*(B)$, M is not the first term of $\mathcal{M}(B)$ unless $B = C_m(x)$. In this case, by Fact B, y is either the first term of Y or y has an immediate predecessor a.

If M has an immediate predecessor N in $\langle \mathcal{M}(B), \leqslant\rangle$, then the maximum a in N^+ is the immediate predecessor of y in $\langle Y, \leqslant\rangle$. So we can assume

$$\mathcal{M}' = \{N \in \mathcal{M}(B) \mid N < M \text{ in } \langle \mathcal{M}(B), \leqslant\rangle\}$$

has no maximum, but, in this case, a_M exists.

In case (iii) either $x = a_M$ or $x = z_M$ or $|M| = 1$. If $x = a_M$, y is the first point of M^+ and $\mathcal{M}'$ has no maximum. By Lemma 6, there is $N \in \mathcal{M}'$ such that, if

$$N^* = \bigcup \{N' \in \mathcal{M}' \mid N < N'\},$$

then $\overline{N^*} = N^* \cup \{x\} \subset U$. Thus, by Fact A, there is $N' > N$ in $\langle \mathcal{M}(B), \leqslant\rangle$ with $f(((N')^*)^+) \subset U$. If $a \in ((N')^*)^+, f((a, y]) \subset U$.

If $x = z_M \neq a_M$ exists and a is maximal in $\{a_M^+\}$, a is the immediate predecessor of y. Otherwise there is no a_M so y is minimal in M^+ and either there is a maximal $N \in \mathcal{M}'$ or $\mathcal{M}' = \emptyset$. If there is a maximal $N \in \mathcal{M}'$ and a is the last term of N^+, then a is the immediate predecessor of y. If $\mathcal{M}' = \emptyset$, $\{x\}$ is the first term of $\mathcal{M}(B)$ and $x \neq q_B$. Thus $B = C_m(x)$. So, by Fact B, either y is the first point of Y of y has as immediate predecessor a in $\langle Y, \leqslant\rangle$.

We have shown that if y is the first point of $\{x\}^+$, either y is the first point of Y or there is $a < y$ in $\langle Y, \leqslant\rangle$ such that $f((a, y]) \subset U$. For the second half of Fact C one assumes that y is the last point of $\{x\}^+$ and absolutely symmetrically one repeats the above proof replacing first by last, greater by less than, a_M by z_M, p_{m0} by p_{m1}, etc., the only difference being that in case (ii) the symmetry begins with y being the last point of D^+ rather than the first. $\square$

Lemma 17. *f is continuous.*

Proof. Suppose $x \in U$ which is clopen in X. We need to prove that if $y \in E(x)$, there is an open interval I_y of $\langle Y, \leqslant\rangle$ with $y \in I_y$ and $f(I_y) \subset U$. We number the cases to make checking easier.

(1) *Suppose $y = x_{n0}$ for some $n \in \omega$ and $C = C_n(x)$.*

(1.a) *Suppose $x = q_D$ for some $D \in \mathcal{D}_\omega(C)$.* For each $t \in \omega$ there is $B_t \in \mathcal{B}_t(C)$ with $D = \bigcap_{t\in\omega} B_t$. Choose $L(t) \in \mathcal{L}_{B_t}$ such that $\overline{B_{t+1}} \subset V_0(L(t))$. Then, if $V = \bigcup_{t\in\omega} V_1(L_t)$, $C \subset (D \cup V)$. Since $x \in D$, $\{x\} = (\overline{V} - V)$, and X is compact, there is $t_0 \in \omega$ with $(\overline{B_{t_0}} - D) \subset U$. For $t \in \omega$, if $Z_t = (\overline{B_t} - B_{t+1})$ and $Z_t^* = \overline{B_t} - D$, by Fact A, there is $t > t_0$ such that $f(B_t^+ - D^+) \subset U$. Since D^+ is a subinterval of B_t^+ in $\langle Y, \leqslant\rangle$, if a is the first point of B_t^+, $f((a, y)) \subset B_t^+ \subset U$.

The first point b of D^+ in $\langle Y, \leqslant\rangle$ is the immediate sucessor of y in $\langle Y, \leqslant\rangle$. So $I_y = (a, b)$ is an open interval of $\langle Y, \leqslant\rangle$, $y \in (a, b)$ and $f(I_y) \subset U$.

(1.b) *Suppose $x = a_M$ where $t \in \omega$, $B \in \mathcal{B}_t(C)$ and $M \in \mathcal{M}^*(B)$.* Let $\mathcal{M}' = \{N < M \text{ in } \langle \mathcal{M}(B), \leqslant\rangle\}$ and for $N \in \mathcal{M}'$ let $N^* = \bigcup\{N' \in \mathcal{M}' \mid N \leqslant N'\}$. By Lemma 6, there is $N_0 \in \mathcal{M}'$ such that $\overline{N_0^*} \cup \{x\} \subset U$. By Fact A, there is $N \in \mathcal{M}'$ such that $f((N^*)^+) \subset U$. If a is the first point of N^+ in $\langle Y, \leqslant\rangle$, (a, y) is an interval of $\langle Y, \leqslant\rangle$ with $f((a, y)) \subset U$.

If b is the first term of M^+, b is the immediate sucessor of y. Thus $y \in I_y = (a, b)$ and $f(I_y) \subset U$.

(2) *Suppose $y = x_{n1}$.* In an exactly symmetric fashion to cases (1.a) and (1.b) one can choose I_y as desired.

(3) *Suppose $y = x_{nr}$ for some $r \geqslant 2$ where $C = C_n(x)$, $t \in \omega$, $M \in \mathcal{M}_t(C)$, $B \in \mathcal{B}_M$ and $x = q_B$.*

(3.a) *$B = B_{r-1}(M)$ and r is odd.* There is a maximal a in B^+ and a is the immediate predecessor of x_{nr} in $\langle Y, \leqslant\rangle$. If there is a lexicographically minimal $\langle n', r'\rangle$ greater than $\langle n, r\rangle$ for which $r' > 2$, then $b = x_{n'r'}$ is the immediate successor of x_{nr} in $\langle Y, \leqslant\rangle$. Otherwise $b = x$ is the immediate successor of x_{nr} in $\langle Y, \leqslant\rangle$. Thus $I_y = (a, b) = \{y\}$ is an open interval of $\langle Y, \leqslant\rangle$ and $f(I_y) \in U$.

(3.b) *$B = B_r(M)$ and r is even.*

(3.b.1) $x \notin \overline{B}$. There is a minimal term b in B^+ and thus b is the immediate successor of x_{nr}.

(3.b.2) $x \in \overline{B}$. For $N \in \mathcal{M}(B)$ let

$$N^* = \bigcup\{N' \in \mathcal{M}(B) \mid N' < N \text{ in } \langle (B), \leqslant\rangle\}.$$

Since $\{x\} = (\overline{N^*} - N^*)$ for all $N \in \mathcal{M}(B)$, there is $N_0 \in \mathcal{M}(B)$ such that $\overline{N^*} \subset U$ (see Lemma 6). By Fact A, there is $N < N_0$ in $\langle \mathcal{M}(B), \leqslant \rangle$ such that $f(N^*)^+) \subset U$. Then, if b is the first point of N^+ in $\langle Y, \leqslant \rangle$, $f((y, b)) \subset U$.

(3.b.3) *There is a lexicographically maximal $\langle n', r' \rangle$ less than $\langle n, r \rangle$ for which $r' > 2$ and $a = x_{n'r'}$ is defined.* Then a is the immediate predecessor of y.

(3.b.4) *Not* (3.b.3) *but there is a maximal $m < n$ such that $R_m(x) \neq \emptyset$; $R_m(x)$ is infinite.* Let N be the term of $\mathcal{M}_m$ with $x \in D_N$. By Corollary 10 there is $u \in \omega$ such that, for all $v > u$ in $R_m(x)$, $B_v(N) \subset U$. By Fact A, there is $w > u$ in $R_m(x)$ such that $f(B_v(N)^+) \subset U$ for all $v \geqslant w$ in $R_m(x)$. If $a = x_{mw}$, $f((a, y)) \subset U$.

(3.b.5) *$\langle n, r \rangle$ is (lexicographically) minimal among $\{\langle n', r' \rangle \mid r' \geqslant 2,\ x_{n'r'}$ is defined$\}$.* In fact, since x_{nr} is defined, there is a maximal $m \in \omega$ for which $C_m(x)$ is defined. Thus, by Fact C, since $y = x_{nr}$ is minimal in $\{x\}^+$, there is either $a < y$ in $\langle Y, \leqslant \rangle$ with $f((a, y)) \subset U$ or y is minimal in all of Y.

In case (3.b) we have either (3.b.1) or (3.b.2) and in each case we chose an interval (y, b) of $\langle Y, \leqslant \rangle$ with $f((y, b)) \subset U$. Also in case (3.b) one has either (3.b.3) or (3.b.4) or (3.b.5). In all of these cases we proved that either y is minimal in $\langle Y, \leqslant \rangle$ or there is $a < y$ in Y with $f((a, y)) \subset U$. Thus $I_y = (a, b)$ or $[y, b)$, depending on the case, is an open interval of $\langle Y, \leqslant \rangle$ containing y with $f(I_y) \subset U$ as desired.

(4) *Suppose $y = x$.*

(4.1) *Suppose there is a lexicographically maximal $\langle n, r \rangle$ for $r > 2$ for which x_{nr} is defined.* Then $a = x_{nr}$ is the immediate predecessor of x.

(4.2) *Suppose there is a maximal $n \in \omega$ for which $R_n(x) \neq \emptyset$ and $R_n(x)$ is infinite.* Let M be the term of $\mathcal{M}_n$ with $x \in D_M$. As in (3.b.4) there is $s' \in R_n(x)$ such that, for all $r' > s'$ in $R_n(x)$, $f(B_{r'}(M)^+) \subset U$. Let $a = x_{nr'}$. Then $f((a, x]) \subset U$.

(4.3) *Suppose that for every $n \in \omega$ there is $D_n \in \mathcal{D}_n$ such that $x \in D_n = D_n(x)$. (So $R_n(x) = \emptyset$ for all $n \in \omega$.)* As proved in Lemma 15, Case (c), for each $n \in \omega$ there is an open U_n with $D_n \subset U_n$ and

$$\{x\} = \bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} D_n.$$

There is $n \in \omega$ with $D_n \subset U_n \subset U$. As pointed out after Fact A there is $n' > n$ in ω such that $f(D_{n'}^+) \subset U$. Let a be the first point of $D_{n'}^+$ and b the last. Then (a, b) is an open interval of $\langle Y, \leqslant \rangle$, $f((a, b)) \subset U$ and $x \in (a, b)$ unless x is a or b. If $x = a$, the first point of $D_{n'}^+$, there is a minimal $m \leqslant n'$ such that x is the first point of D_m^+. Then as proved in Fact C, $y = x$ is the first point of $\langle Y, \leqslant \rangle$ or x has an immediate predecessor. If $x = b$, either x is the last point of $\langle Y, \leqslant \rangle$ or x has an immediate successor. In any case for *some* a and b, $I_y = (a, b)$ or $[x, b)$ or $(a, x]$ is an open interval of $\langle Y, \leqslant \rangle$ to which y belongs and $f(I_y) \subset U$.

(4.4) *Suppose there is a maximal $m \in \omega$ for which $C_m(x)$ is defined.* (Equivalently, not (4.3).) Since x is always the last point of $\{x\}^+$, by Fact C either $y = x$ is the last point of Y or there is $b > x$ in $\langle Y, \leqslant \rangle$ with $f((x, b)) \subset U$. If $R_n(x) = \emptyset$ for all $n \in \omega$, then x is the first point of $\{x\}^+$ and by Fact C either x is the first point of Y or there is $a < x$ in $\langle Y, \leqslant \rangle$ with $f((a, x)) \subset U$. If $R_n(x) \neq \emptyset$ we have either Case (4.1) or

Case (4.2) and in these cases there is $a < x$ in $\langle Y, \leqslant \rangle$ with $f((a, x)) \subset U$. Thus in all cases, defining $I_y = (a, b)$ unless x is the first point of Y in which case we take $I_y = [x, b)$, or x is the last point of Y in which case we take $I_y = (a, x]$, we have $x \in I_y$ and $f(I_y) \subset U$.

This concludes the proof of our theorem. □

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